A spatial population model in separating time windows

Diplomarbeit von

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betreut von

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Abstract

A stochastic and discrete analogue of the logistic differential equation is considered in a spatial context. Individuals live on a finite set of separated locations and perform, after exponentially distributed waiting times, the following actions: Single individuals give birth to new individuals that appear on the parent location; pairs of individuals on the same location coalesce; and individuals migrate between locations. The migration is of mean field type. This is modelled as a pure jump type Markov process with linear birth and quadratic death rates, which justifies the comparison with the logistic equation.

This work studies the evolution towards equilibrium when the system is initialized with one particle on one location. Two different time windows are considered: In the first, the number of inhabited locations grows at an exponential speed; this is proven by coupling the evolution to a Crump Mode Jagers branching process. The second time window describes the filling of space until an equilibrium is attained; now, the increasing probability to migrate to an already inhabited location attenuates the growth. Convergence in this second time window of the time shifted and normalized process is proven via a generator calculation, when the number of locations is considered as a parameter and sent to infinity. The limiting process is nonlinear and deterministic except for a random time change that can be interpreted as the randomness that has been collected in the first time window. In particular, the proportion of inhabited colonies itself satisfies a logistic differential equation with time-dependent coefficients and random initial state. The time windows separate because the mentioned filling of space happens later when the number of locations is increased.

This work is based on Chapter 7 of the text On the effects of migration in spatial Fleming-Viot models with selection and mutation by D. A. Dawson and A. Greven.

Contents

1 Introduction

1.1 The logistic equation

The "strong and constantly operating check on population from the difficulty of subsistence" prohibits, in the words of Thomas R. Malthus, the unlimited growth of populations. His Essay on the Principle of Population, published in 1798, strongly influenced the Belgian Mathematician Pierre-François Verhulst. In the year 1838, he introduced in his Notice sur la loi que la population suit dans son accroissement what is today known as the *logistic equation* (quoted from the translation in [VZ1975]):

> If p is the population, then dp is an infinitesimally small increase that it receives in a very short period of time dt. If the population increases by geometric progression, we would have the equation $dp/dt = mp$. However, as the rate of population growth is slowed by the very increase in the number of inhabitants, we must subtract from mp an unknown of p, so that the formula to be integrated can be written as
>
> $$\frac{dp}{dt} = mp - \phi(p)\,. \tag{1.1}$$
>
> The simplest hypothesis that can be made on the form of the function ϕ is to suppose that $\phi(p) = np^2$.

In the present work, the deterministic logistic equation (1.1) is replaced by a stochastic and discrete analogue: A population of individuals is considered, where births of individuals happen randomly at a rate that is linear and deaths happen at a rate that is quadratic in the number of individuals.

Additionally, a geographic structure is introduced. Initially, the population is located at a single location; in the course of time, individuals migrate and settle new populations on unoccupied locations. It is the goal of the present work to understand the evolution both on small and on large scales, i. e. the population on a fixed location as well as the settlement of previously empty locations. It will turn out that the proportion of inhabited locations evolves in time very similar to the solution of the logistic equation; the present stochastic particle model follows thus logistic evolution rules on two different scales.

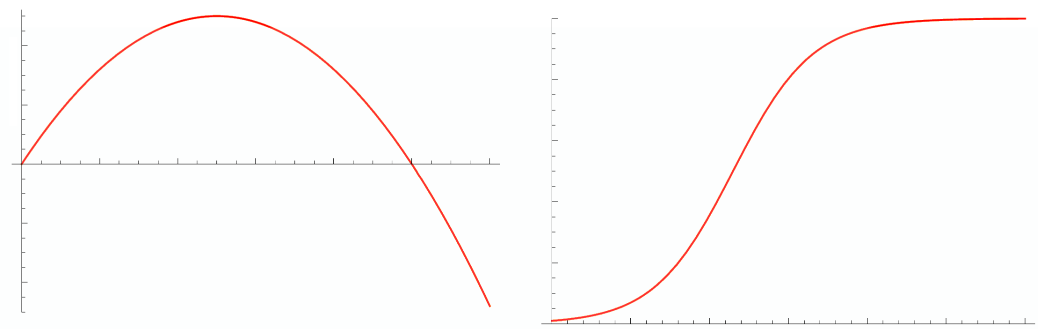


Figure 1.1: The left picture is a plot of the force that the evolution p is exposed to, i. e. of the right hand side of equation (1.1) as a function of p. The right picture shows, for a given small initial value $p(0)$, the evolution of the solution to (1.1) as a function of t.

1.2 Brief description and organization of the diploma thesis

The diploma thesis describes, in the framework of pure jump type Markov processes, a population of individuals on a finite or infinite set of locations. The individuals are in the following called *particles*, the locations *colonies*. The particles evolve according to the following rules:

- Particles give birth to new particles. These are located at the same colony.
- Pairs of particles at the same colony coalesce. This implies that particles are exposed to a death rate that is proportional to the squared number of particles at the respective colony.
- Particles migrate. If there are only finitely many colonies, the migration destination is chosen uniformly at random amongst all locations. Otherwise, a migration always leads to a free, previously unoccupied colony.

We are primarily interested in the case when the number of colonies is finite such that circular migration is possible (that is, an emigrant might come back); this model is called

the N Colony System,

where N denotes the number of colonies. But also the simpler case of an infinite colony space is considered in which migrations lead always to new colonies. This model is called

the Collision Free System.

The overall goal is to describe the filling of space via migrations and related functionals in the limit when the finite number of islands goes to infinity. It turns out that the limiting process is qualitatively different from the process that begins with infinitely many colonies in the first place. The reason is that in each finite space model finally a stable equilibrium is reached that fills the whole space evenly, a behaviour which cannot be found in the infinite island model.

This space filling happens later and later when the number of islands increases. This also implies that, loosely speaking, the time window of observation must move to infinity; otherwise, the interesting characteristics of the finite space models disappear to the right and are not visible anymore in the limit.

Hence, these are the properties of the population model that are under examination in this diploma thesis:

- Each population quickly reaches a local equilibrium and, after some time, also a global equilibrium that fills the whole colony space.
- In spite of the fact that the fluctuations of the populations on fixed islands show highly random behaviour, the spreading into space follows asymptotically deterministic evolution equations. This is caused by a law of large numbers that comes into effect when many particles and islands act simultaneously. Here, asymptotically means both with increasing time and with increasing space.
- The time windows of interest separate: From $t = 0$ onwards, the evolution of the finite space model resembles more and more that of the infinite space model when the space parameter N increases. From some time $t = t(N)$ (which increases with N) onwards and normalized adequately, the evolution describes the asymptotically deterministic filling of space.

The diploma thesis is organized as follows:

- This first chapter introduces the key quantities and summarizes the mathematical results. In the Sections 1.3 and 1.4, the models as well as the functionals that will be examined later are defined. This is followed in Section 1.5 by simulation plots in order to provide some intuition on these systems. In Section 1.6, the results that are obtained in this work are presented, first verbally and then rigorously with cross-references to later chapters where the proofs are given.

- The remaining text is divided into three parts: Part I deals with the first time window in which the finite model converges to the infinite model, Part II tries to bridge between both time windows that separate when N gets large, and Part III examines the second time window in which the finite models show an autonomous evolution.

The present work fits into the literature as follows: A continuous mass diffusion limit of the infinite island model is studied by M. Hutzenthaler in his work [MH2009], where the model carries the name The Virgin Island Model. Furthermore, the present discrete model itself arises as the dual to a certain diffusion in the study of a continuous mass model in which the combined effects of migrations, mutations and selection are studied. This is done in the work On the effects of migration in spatial Fleming-Viot models with selection and mutation by D. A. Dawson and A. Greven, cf. [DG2010]. The connection between the model of Dawson and Greven and the present model is summarized below in Section 1.7.

The work of Dawson and Greven also is the foundation of the present work. The main ideas and proof techniques are taken from [DG2010], while some details have been worked out by the present author. This holds in particular for Chapter 2.5 and Chapters 4-6.

1.3 Definition of the models

The definitions of the models are taken from Chapter 7 of [DG2010]. The system describes particles living on a certain number of colonies; this number is denoted with N, N possibly infinite. The particles' evolution mechanism is given by branching, coalescence and migration. In this chapter we define the dynamics and a framework of notation.

First, we give a verbal definition of the process. This definition mainly serves to introduce the constants of birth, migration and coalescence rates. A more rigorous construction will be given in Chapter 3 where a coupling is introduced that ties together the trajectories of the systems for different values of N. Throughout, assume that

$$(\Omega,\ \mathcal{A},\ \mathbb{P}) \tag{1.2}$$

is a probability space that is rich enough to accommodate all the introduced quantities.

Definition 1.1 (The N Colony System)**.**
Let $N \in \mathbb{N}$ and s, d, $c > 0$ be given constants. The particle system

$$\zeta^N = (\zeta_i^N(t)\ :\ i = 1,\ ...,\ N,\ t \geq 0) \tag{1.3}$$

on colony space $\{1,\ ...,\ N\}$ is called N Colony System. *It evolves according to the following mechanisms that act independently of each other:*

- *Each particle* gives birth *to a new particle, located at the same colony, at rate s.*
- *Particles* migrate *between colonies at rate c; the travel destination is chosen uniformly at random amongst the N colonies. In particular, migration to the own colony happens with probability N^{-1}.*
- *Each particle dies with rate $\frac{d}{2}(n-1)$, where n is the number of particles present at the current location. Another way to express this is to say that at a fixed location, each pair of particles* coalesces *with rate d.*

The system starts at time $t = 0$ with one particle at the colony with index 1. Particles do not carry any information besides their location; this implies that it is sufficient to record in each colony the number of particles present. In this formulation, ζ^N is a pure jump type Markov process with state space

$$\mathbb{S}^N = \mathbb{N}_0^{\{1,\ ...,\ N\}}\ , \tag{1.4}$$

such that

$$\zeta_i^N(t) \tag{1.5}$$

denotes the number of particles in the i^{th} *colony at time* t*.*

Definition 1.2 (Terminology)**.**
The following terminology is used throughout the text:

- *The particles that are alone on their respective colonies are called* single*. A* single particle migration *happens thus when a single particle migrates.*
- *A migrating particle* collides *if it migrates to a currently occupied colony. Such an event is referred to as a* collision*.*

Note that when a single particle migration leads to a previously unoccupied colony, it changes the configuration only by a permutation of the colony indices. In particular, it leaves the number of occupied colonies constant. Similarly, the number of occupied colonies decreases only by a colliding single particle migration.

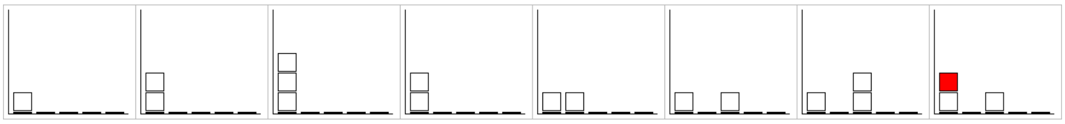


Figure 1.2: An example of the evolution in time of the N Colony System on five colonies, in discrete time; each of the eight pictures shows one time step (the continuous time process stays a random amount of time in the various states). We observe two successive births followed by the death of a particle, then two migrations (one of which is a single particle migration), then a birth followed by a migration. This last migration leads to a previously occupied colony; this is a so-called collision. The collided particle is coloured red; such colours are not part of the original model but will be used later in an enhanced system to keep record of the impacts of collisions.

Next, we define the candidate for the limiting system as $N \to \infty$ in the initial time window starting at $t = 0$.

Definition 1.3 (The Collision Free System)**.**
The particle system

$$\zeta = (\zeta_i(t) \ : \ i \in \mathbb{N}, \ t \geq 0) \tag{1.6}$$

on colony space $\mathbb{N}$ *is called* Collision Free System*. Here, particles act according to the same mechanisms as in Definition 1.1 except for migration: The modified rule is that each migration leads to a previously unoccupied colony; this is always the colony carrying the greatest index of any occupied colony plus one.*

Counting again the number of particles per colony, ζ *is then a pure jump type Markov process with the still countable state space*

$$\mathbb{S} = \bigoplus_{N=1}^{\infty} \mathbb{S}^N = \left\{ s \in \mathbb{N}_0^{\mathbb{N}} \ : \ \textit{there exists an } m_0 \textit{ such that } s(m) = 0 \textit{ for all } m \geq m_0 \right\} . \tag{1.7}$$

Remark 1.4. *The rule according to which a new colony is selected in the course of a migration ensures that that the movement of particles across the colonies always goes "to the right"; this will simplify later the coupling to the N Colony System. Another way to describe this travel target is "the first colony where previously no particle has been", but the formulation in Definition 1.3 clarifies that the dynamics are still Markovian.*

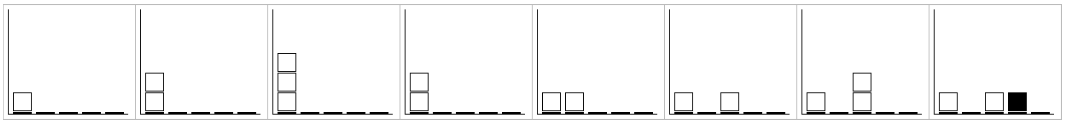


Figure 1.3: An example of the evolution in time of the Collision Free System, zoomed in on the first five colonies. Again, only the discrete time jump process is plotted. Up to the time of the first collision, the evolution follows the same rules as those of the N Colony System; both systems can thus be realized on the same probability space such that the paths are identical up to this separation time. In the last time step, both systems separate; the particle that would collide in the N Colony System is coloured black. By comparing the offspring of the black particle with the offspring of the red particle, we have a tool to quantify the difference between both systems. These pictures already contain the idea of coupling that will be introduced in Chapter 3.

1.4 Definition of the functionals

In order to formulate the mathematical results which will be presented in Section 1.6, we need further notation. The following functionals are motivated by simulations in Section 1.5.

We define first the statistics Ψ^N, Ψ of the processes counting the number of colonies carrying a given number of particles. This is followed below by the quantities K^N, K that count the number of occupied colonies and Π^N, Π that count the total number of particles in the system. Finally, we introduce the hitting times $T_\cdot$, $T_\cdot^N$ that give the times when the numbers of occupied colonies reach a given level.

Definition 1.5 (The statistics of the processes)**.**
Let for all $t \geq 0$

$$\Psi^N(t,\,\cdot),\ \Psi(t,\,\cdot) \in \mathcal{M}_{fin}(\mathbb{N}) \tag{1.8}$$

denote the statistics *(or* empirical size distributions*) of the processes ζ^N, ζ respectively, defined for $j \in \mathbb{N}$ via*

$$\Psi^N(t,\,j) \;=\; \textit{number of colonies that carry } j \textit{ particles at time } t \textit{ in the } N \textit{ Colony System,} \tag{1.9}$$

and

$$\Psi(t,\,j) \;=\; \textit{number of colonies that carry } j \textit{ particles at time } t \textit{ in the Collision Free System.} \tag{1.10}$$

More formally, these quantities can be expressed as follows:

$$\begin{aligned} \Psi^N(t,\,j) &= \sum_{k=1}^{N} 1_{\{\zeta_k^N(t)=j\}}\,, \\ \Psi(t,\,j) &= \sum_{k=1}^{\infty} 1_{\{\zeta_k(t)=j\}}\,. \end{aligned} \tag{1.11}$$

Remark 1.6. *The statistics follow themselves Markovian evolution rules that can be written down in a more compressed form than those of the original systems. This will be exploited in Part III for generator calculations. But the processes are not equivalent: The original systems ζ^N, ζ carry more information than the statistics Ψ^N, Ψ respectively, because, for instance, if we observe two transitions*

$$\psi_1 \mapsto \psi_2 \mapsto \psi_3 \tag{1.12}$$

in the statistics that are both due to migrations, we cannot decide whether it was the same *colony where a particle immigrated to; or if the second transition was an immigration to a colony with merely the same occupation number.*

We now introduce the quantities K^N, K and Π^N, Π that count inhabited colonies and particles respectively.

Definition 1.7 (Colonies and inhabitants)**.**
We need the following functionals:

- *Let*
$$K^N(t),\ K(t) \tag{1.13}$$
denote the number of inhabited colonies *in the respective systems, i. e.*
$$\begin{aligned} K^N(t) &= \sum_{k=1}^{N} 1_{\{\zeta_k^N(t)\neq 0\}} = \sum_{j=1}^{\infty} \Psi^N(t, j)\,, \\ K(t) &= \sum_{k=1}^{\infty} 1_{\{\zeta_k(t)\neq 0\}} = \sum_{j=1}^{\infty} \Psi(t, j)\,. \end{aligned} \tag{1.14}$$

- *Let furthermore*
$$\Pi^N(t),\ \Pi(t) \tag{1.15}$$
denote the total number of particles *in the respective systems, i. e.*
$$\begin{aligned} \Pi^N(t) &= \sum_{k=1}^{N} \zeta_k^N(t) = \sum_{j=1}^{\infty} j\Psi^N(t, j)\,, \\ \Pi(t) &= \sum_{k=1}^{\infty} \zeta_k(t) = \sum_{j=1}^{\infty} j\Psi(t, j)\,. \end{aligned} \tag{1.16}$$

We finally come to the hitting times of the functionals K^N, K.

Definition 1.8 (Hitting times)**.**
Define for $M \in \mathbb{R}_+$ the hitting times
$$\begin{aligned} T_M^N &= \inf\{t \geq 0 \ :\ K^N(t) \geq M\}\,, \\ T_M &= \inf\{t \geq 0 \ :\ K(t) \geq M\}\,. \end{aligned} \tag{1.17}$$

Remark 1.9. *By definition,*
$$T_M^N = T_{\lceil M\rceil}^N\,, \tag{1.18}$$
where $\lceil M\rceil$ denotes the smallest integer that is larger than or equal to M. This gives the time when M populated colonies are reached; but due to single particle migrations, this is not identical to the time when the colony indexed with M is invaded. Most often used will be the choices $M(N) = \log N$ and $M(N) = \epsilon N$ for previously chosen $\epsilon > 0$. These times hint when the first time window is left and when the second begins:

- *Up to time $T_{\log N}^N$, a collision occurs with probability* $\log N/N = o(1)$*; hence, the probability that no collision occurred is about*
$$\frac{(\log N)^2}{N} = o(1)\,, \qquad (N \to \infty)\,. \tag{1.19}$$
We assign the time $T_{\log N}$ therefore to the initial collision free time window. Of course, the choice $\log N$ *is rather arbitrary; another reasonable choice would be $N^{\frac{1}{2}-\beta}$ for some $\beta > 0$. It is however noteworthy that* (1.19) *only implies that the Collision Free and the N Colony System are close in probability; but we cannot invoke the Borel Cantelli Lemma to argue that, in the time interval $[0, T_{\log N}^N]$, almost surely no collision occurs for all large N.*

- *From time $T_{\epsilon N}^N$ onwards, the collision probability is at least $\epsilon N/N = \epsilon$. Collisions now become notable and it is no longer reasonable for the N Colony System to be approximated by the Collision Free System. Hence, we put the beginning of the second time window at time $T_{\epsilon N}^N$ for some small ϵ, understanding that this is not a strict definition.*

1.5 Simulations

In this section, we show simulation plots in order to provide some intuition on the systems. The emphasis is on qualitative understanding so that numerical values have been omitted on the axis. Throughout the diploma thesis, a simulation of the particle systems ζ^N, ζ in their multicolour coupling is plotted in the footer of the pages. This coupling follows the idea that has been sketched in Figures 1.2 and 1.3 and that is fully introduced in Chapter 3. For a quick understanding, it is sufficient to comprehend that the white particles together with the red give a version of ζ^N while the white and black particles give a version of ζ.

We first consider a simulation of K^N. This is shown in Figure 1.4.

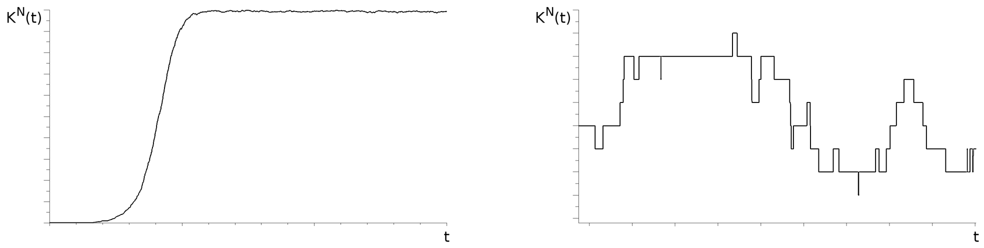


Figure 1.4: Plot of the simulation data of the functional $K^N(t)$:
a) The time window is [0, 150]. Although the space quickly fills with particles, there is still fluctuation around some equilibrium value.
b) The time window is [73, 78] and the y axis ranges from 990 to 1000. The fact that there are so many steps downwards indicates that there are still many colonies carrying only one particle. The parameters of the simulation are $N = 1000$, $s = 1$, $d = 0.2$ and $c = 0.5$.

Secondly, the statistics Ψ are simulated. The plot of the normalized numbers is shown in Figure 1.5.

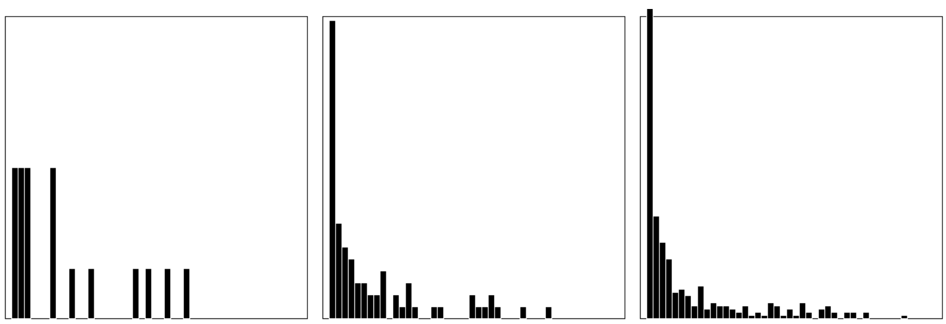
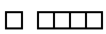


Figure 1.5: The normalized statistics $K(t)^{-1}\Psi(t, j)$, plotted at step 500, 2000, 8000 respectively of the embedded jump chain. It is shown in Chapter 2 that this normalized object converges to some stable distribution Ψ_∞ in the limit $t \to \infty$. The x-axis ranges from $j = 1$ to $j = 50$, the y-axis from 0 to 0.33. The parameters are $s = 15$, $d = 0.5$, $c = 1$.

One aspect that has not been mentioned yet is the following: When running the simulation of the N Colony System several times, always the same curves appear, but shifted sidewards by a random amount of time. This is shown in Figures 1.6 and 1.7.



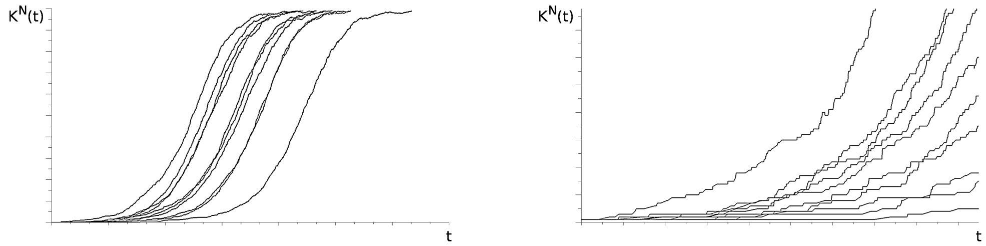


Figure 1.6: a) 10 realizations of $K^N(t)$.
b) The early time window "where randomness comes in". In the beginning, few collisions occur and the trajectories are, at least in distribution, close to those of the Collision Free System. In particular, there is almost no decrease of $K^N(t)$. (The parameters are the same as in Figure 1.4.)

The interpretation is the following: As soon as the system enters a regime where many particles and colonies act simultaneously, a law of large numbers takes over. Any functional that, loosely speaking, depends on large scales becomes deterministic. However, the evolution of any fixed colony is highly random, even in equilibrium. This small scale randomness leads to the random time shift: In an initial time window, the dynamics are governed by only a few exponential clocks which behave differently in each realization.

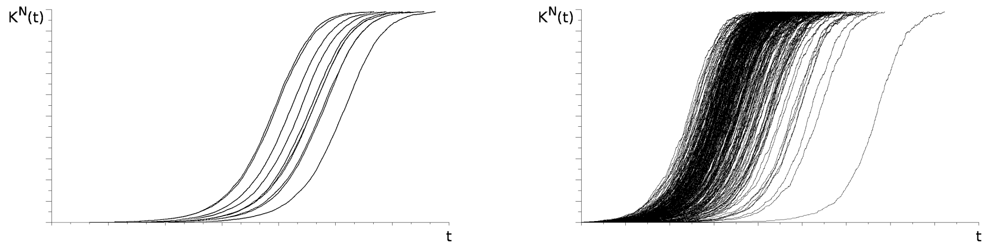


Figure 1.7: a) 10 sample paths of $K^N(t)$ with $N = 10000$. Since "randomness comes in" in an early time window where almost no collisions occur, this randomness does not depend on N. Hence, we expect the distribution of the random time shift to be roughly the same for all N.
b) 300 sample paths of $K^N(t)$ with $N = 1000$. This indicates that the random time shift has a rather concentrated distribution with light tails as we would expect it for a sum of exponentials.

Remark 1.10. *It is intriguing that also the evolution of $K^N(t)$ as seen in Figure 1.7 has an S-like shape like the solution to the logistic equation (cf. Figure 1.1). An heuristic explanation is the following: In textbooks, the logistic equation* (1.1) *is most often rewritten in the form*

$$\tfrac{dp}{dt} = mp(1 - \frac{p}{M}), \tag{1.20}$$

where M is the so called carrying capacity of the environment. In the words of D. Bradley, M is "the maximum number of niches that the ecosystem can support". He points out in his article [DB2001] that a simple way to find a solution is via the substitution

$$r = \frac{M-p}{p}, \quad \text{which leads to} \quad \tfrac{dr}{dt} = -rm\,. \tag{1.21}$$

Bradley interprets r as "the dimensionless ratio of available or vacant niches to niches currently occupied". His substitution makes it clear that this ratio decays exponentially. The vacant and populated niches now can be compared with empty and inhabited colonies. The invasion rate of new colonies is linear in the number of populated colonies; and this number *is bounded by the carrying capacity N, the number of possible niches. It is thus reasonable to obtain similar curves. In Theorem 1.17 below, it is stated that the quantity $K^N(t)$ indeed satisfies a logistic differential equation with time dependent coefficients, when taking the limit $N \to \infty$ in an appropriate sense.*

1.6 Results

The overall goal is to understand in a mathematically rigorous way the evolution of the trajectory $t \mapsto K^N(t)$ as seen in Figures 1.4, 1.6 and 1.7, and more generally the evolution of $t \mapsto \Psi^N(t)$. This understanding can be summarized as follows:

1. The initial time window, starting at time $t = 0$:
   (a) The Collision Free System follows simple exponential growth rules.
   (b) Starting at $t = 0$, the N Colony System converges towards the Collision Free System in the limit $N \to \infty$.

   These results are obtained in Part I.

2. Between the time windows:
   (a) For some small fixed $\epsilon > 0$, both systems evolve closely but with quantifiable distance up to time $T^N_{\epsilon N}$ (the time when about ϵN colonies are inhabited).
   (b) Being already in the regime of large numbers, both evolutions become on large scales deterministic.

   These results are obtained in Part II.

3. The second time window:
   (a) For fixed N, the N Colony System converges in the limit $t \to \infty$ towards an equilibrium that can be specified precisely.
   (b) Starting at time $\mathbb{E}\left[T^N_{\epsilon N}\right]$ and normalized adequately, the N Colony System converges pathwise towards a deterministic evolution with random time shift in the limit $N \to \infty$. This shift expresses the randomness that has been collected in the initial time window.

   These results are obtained in Part III.

In the rest of this introduction, these statements are made precise.

1.6.1 The initial time window

The following theorem yields a profound understanding of the Collision Free System.

Theorem 1.11 (Exponential growth)**.**
The Collision Free System ζ grows exponentially: There exists some $\alpha > 0$ and some random variable W such that

$$\lim_{t\to\infty} \frac{K(t)}{\exp(\alpha t)} = W\,, \tag{1.22}$$

and this convergence holds almost surely, in L^1 and in mean square. The random variable W has finite mean and variance and is almost surely strictly positive.

Outline of proof. This is shown in Theorem 2.1 with the tool of generalized age-depending branching processes (so called Crump Mode Jagers processes). □

This exponential growth is easily understood by the following self-similarity property of the Collision Free System: For each fixed colony index i, the colony together with its offspring evolves, when time is measured from its invasion time onwards, exactly like the whole process ζ itself. Due to the absence of collisions, the "invasion graph" that is obtained by collecting all migration data has a tree-like shape; all forks evolve thus independently of each other. The global picture can thus be compared with the growth of a Galton Watson tree which, in the supercritical case, is also known to be exponential. Clearly, the tree structure as well as the independencies get lost when considering the N Colony System due to circular migrations, and this is why the latter is a more intricate object to study.

As already pointed out, Ψ^N lies close to Ψ for large N and small t; this is the subject of the second theorem below. We need to single out irrelevant discrepancies that can be removed by a relabelling of colonies.

Definition 1.12 (Equivalence of configurations).
Let $N \in \mathbb{N}$ be fixed. Two configurations

$$\tilde{\zeta}_1,\ \tilde{\zeta}_2 \in \mathbb{S}^N \tag{1.23}$$

are said to be label-equivalent *if there exists a permutation η on the numbers $\{1, ..., N\}$ such that*

$$\tilde{\zeta}_1 = \tilde{\zeta}_2 \circ \eta\,. \tag{1.24}$$

A label-equivalent modification *of the process $(\zeta^N(t))_{t\geq 0}$ is a $\mathbb{S}^N$ valued process*

$$(\tilde{\zeta}^N(t))_{t\geq 0} \tag{1.25}$$

that is defined on the same probability space such that $\tilde{\zeta}^N(t)$ is label-equivalent to $\zeta^N(t)$ for all $t \geq 0$.

Theorem 1.13 (Convergence in the first time window).
The Markov processes ζ and $\{\zeta^N : N \in \mathbb{N}\}$ can be realized on the same probability space such that random relabellings of ζ^N converge for $N \to \infty$ almost surely uniformly on compacta towards ζ. More precisely, for all $N \in \mathbb{N}$ there exists a label equivalent modification $\tilde{\zeta}^N$ of ζ^N, and for these processes the following holds: For all time horizons $T \in \mathbb{R}$, $T > 0$ there exists a $N_0 \in \mathbb{N}$ such that

$$\sup_{t\leq T}\ \sup_{i\leq N} \left|\tilde{\zeta}_i^N(t) - \zeta(t)\right| = 0 \tag{1.26}$$

almost surely for all $N \geq N_0$. In particular,

$$\Psi^N \overset{N\to\infty}{\Longrightarrow} \Psi \tag{1.27}$$

on the path space

$$D([0,\ \infty),\ \mathcal{M}_{fin}(\mathbb{N}))\,. \tag{1.28}$$

Outline of proof. The coupling will be introduced in Chapter 3 and the claim is proven in Theorem 3.6. The idea is that in Figures 1.2 and 1.3 the first appearance of a red or black particle moves to the right when N increases. □

This is a seemingly inconsistent statement, because how can something converge pathwise when the paths are known to separate eventually? The answer is that both the notion of convergence on compacta as well as the convergence with respect to the Skorohod metric neglect deviations that happen far to the right.

1.6.2 Between the time windows

The separation of the trajectories of both systems must certainly happen before time T_N when the Collision Free System reaches N populated colonies. It is reasonable to expect this separation at time $T^N_{\epsilon N}$ for small $\epsilon > 0$ when collisions begin to appear with probability ϵ. The following result gives bounds on the approximation quality up to this time.

Theorem 1.14 (Distance of trajectories)**.**
For small $\epsilon > 0$, the N Colony System and the Collision Free System evolve closely up to time $T_{\epsilon N}$. More precisely, there exists a constant C that does not depend on N nor on ϵ such that, in the correct coupling and if ϵ is sufficiently close to 0,

$$0 \leq \limsup_{N\to\infty} \frac{K(T_{\epsilon N}) - K^N(T_{\epsilon N})}{\epsilon N} \leq C\epsilon\,, \tag{1.29}$$

almost surely. For small ϵ, this can be extended to the time $T^N_{\epsilon N}$ as follows:

$$\limsup_{N\to\infty} \frac{K(T^N_{\epsilon N}) - K^N(T^N_{\epsilon N})}{\epsilon N} \leq \frac{\tilde\epsilon - \epsilon}{\epsilon}\,, \tag{1.30}$$

where $\tilde\epsilon$ is the smallest solution to the equation

$$\tilde\epsilon(1 - C\tilde\epsilon) = \epsilon\,. \tag{1.31}$$

Outline of proof. The coupling is the same that has been used before, and the proof is carried out in Chapter 4 (Theorem 4.1 and Corollary 4.14). The strategy of proof is to show that, in Figures 1.2 and 1.3, the future offspring of the red particle is always smaller than that of the black particle. This leaves the task to find bounds on the black population. Due to the simpler structure of the Collision Free System, this can (almost) be done by mere counting. □

Theorem 1.11 states that as soon as an initial time window is left, say from time $T_{\log N}$ onwards, the functional $K(t)$ evolves deterministically. Theorem 1.14 states that $K^N(t)$ stays close to $K(t)$ up to time $T^N_{\epsilon N}$ for some small ϵ. The next step now is to ask for the randomness in the trajectory of $K^N(t)$.

A measure on this randomness is the fluctuation in the waiting time between the crossings of two given barriers. The following result shows that the variance of this waiting time, conditioned on the precise entrance point into the first barrier, vanishes for $N \to \infty$.

Theorem 1.15 (Deterministic evolution between $\log N$ and ϵN)**.**
For small $\epsilon > 0$, the following holds:

$$\begin{aligned} \lim_{n\to\infty} \sup_{\psi} \mathbb{V}ar(T_{\epsilon N} - T_{\log N} \mid \Psi(T_{\log N}) = \psi) &= 0\,, \\ \lim_{n\to\infty} \sup_{\psi} \mathbb{V}ar\left(T^N_{\epsilon N} - T^N_{\log N} \mid \Psi^N(T^N_{\log N}) = \psi\right) &= 0\,. \end{aligned} \tag{1.32}$$

Here, the supremum is taken over all admissible configurations ψ, i e. any ψ satisfying $\psi(k) \in \mathbb{N}_0$ and

$$\sum_{k\geq 1} \psi(k) = \lceil \log N \rceil\,. \tag{1.33}$$

Moreover, the hitting times also converge almost surely in the case of the Collision Free System:

$$\limsup_{N\to\infty} \left| T_{\epsilon N} - T_{\log N} - \frac{1}{\alpha} \log \frac{\epsilon N}{\log N} \right| = 0\,. \tag{1.34}$$

Outline of proof. The almost sure convergence follows from inversion of Theorem 1.11. The variance is calculated by approximately solving a certain set of recurrence equations (a discrete Poisson equation); this is done in Theorem 5.2 in Chapter 5. More precisely, using various simplifications, the original infinitely dimensional system is reduced to an asymmetric random walk on $\mathbb{Z}$. The moments of the hitting times for the latter are then calculated explicitly. □

Unfortunately, this last statement is not as strong as it looks at first glance; in particular, it does not allow to conclude that $T^N_{\epsilon N} - T^N_{\log N}$ converges in probability (or equivalently, in distribution) towards a constant.

1.6.3 The second time window

For fixed N, the system finally reaches an equilibrium that fills the whole space evenly. The equilibrium distribution is surprisingly simple.

Proposition 1.16 (The equilibrium)**.**
For $N \in \mathbb{N}$, the equilibrium distribution π^N of the N Colony System on the state space

$$\mathbb{S}^N = \mathbb{N}_0^N \tag{1.35}$$

is given by

$$\pi^N(k_1, ..., k_N) = \prod_{m=1}^{N} \pi^1(k_m), \qquad (k_1, ..., k_N) \in \mathbb{S}^N, \tag{1.36}$$

where π^1 denotes the probability density function of the Poisson distribution with parameter $d^{-1}2s$:

$$\pi^1(k) = \exp\left(-\frac{2s}{d}\right)\left(\frac{2s}{d}\right)^k \frac{1}{k!}. \tag{1.37}$$

Outline of proof. This is done in Proposition 6.1 by checking the detailed balance equations. The constant c of migration does not anymore appear because the probability flux between colonies cancels out in equilibrium. □

We now turn to the evolution towards the equilibrium. From time $T^N_{\epsilon N}$ onwards, the N Colony System separates more and more from the Collision Free System, because collisions begin to play a decisive role. The collision probability increases from ϵ towards 1 in the course of time. When N increases, a law of large numbers takes over; the resulting trajectory is thus asymptotically deterministic.

Theorem 1.17 (Convergence in the second time window)**.**
For any $t_0 \in \mathbb{R}$, there exists an $\mathcal{M}_{\le 1}(\mathbb{N})$ valued continuous process $(\Phi(t))_{t\in\mathbb{R}}$ such that

$$\left(\frac{1}{N}\Psi^N((\frac{1}{\alpha}\log N + t_0 + t) \vee 0)\right)_{t\in\mathbb{R}} \Rightarrow (\Phi(t))_{t\in\mathbb{R}} \tag{1.38}$$

on the path space $D(\mathbb{R}, \mathcal{M}_{\le 1}(\mathbb{N}))$. The process Φ has the following properties:

1. *The process is deterministic up to a random time shift, i. e.*

$$(\Phi(\tau(\epsilon) + t))_{t\in\mathbb{R}} \tag{1.39}$$

is a deterministic process, when $\tau(\epsilon)$ denotes the first passage time of $\Phi(t, \mathbb{N})$ at level $\epsilon \in (0, 1)$.

2. *This random time shift is only caused by the randomness collected in the first collision free time window. More precisely, there exists a deterministic function* $\tilde{\tau}(\epsilon)$ *such that*

$$\tau(\epsilon) \stackrel{d}{=} \tilde{\tau}(\epsilon) + \frac{1}{\alpha} \log \frac{\epsilon}{W}\,, \tag{1.40}$$

where the summand $\alpha^{-1} \log \epsilon W^{-1}$ *can be identified with the limit of the normalized hitting times* $T_{\epsilon N} - \alpha^{-1} \log N$ *of the Collision Free System.*

3. *Given any initial point* Φ_0*, the proportion of occupied colonies* $\Phi(t, \mathbb{N})$ *satisfies a logistic differential equation with time dependent coefficients:*

$$\frac{d}{dt}\Phi(t, \mathbb{N}) = \alpha(\Phi(t, \cdot))\Phi(t, \mathbb{N}) \left(1 - \left(1 + \frac{\gamma(\Phi(t, \cdot))}{\alpha(\Phi(t, \cdot))}\right) \Phi(t, \mathbb{N})\right)\,, \tag{1.41}$$

where

$$\gamma(\Phi(t, \cdot)) = c\Phi(t, 1), \qquad \alpha(\Phi(t, \cdot)) = c \sum_{k \geq 2} k\Phi(t, k)\,. \tag{1.42}$$

Also, the components $\{\Phi(t, j)\,,\, j \in \mathbb{N}\}$ *satisfy a coupled system of differential equations that can be specified explicitly.*

4. *If* $t_0 = 0$*, then*

$$\lim_{t \to -\infty} \frac{\Phi(t, \mathbb{N})}{\exp(\alpha t)} \stackrel{d}{=} W\,, \tag{1.43}$$

where W *is the growth variable that was specified in Theorem 1.11.*

Outline of proof. This follows from a generator calculation and is carried out in Chapter 7. □

The first assertion states that the process is deterministic which we attribute to law of large number effects. The second states that all remaining randomness is caused by the initial regime of small numbers. The third assertion explains why the plot of K^N has the familiar S shape that is known from the solution of the logistic differential equation. Finally, the fourth assertion states roughly that, while time $t = 0$ is no more visible in the limit $N \to \infty$ when we shift the process to $\mathbb{E}\left[T^N_{\epsilon N}\right]$, we see at least the collision free time window which we have examined earlier.

1.7 The role of the model in the work of Dawson and Greven

The N Colony System arises in the article [DG2010] of Dawson and Greven as the dual system of an $\mathbb{R}^N$ valued diffusion. We sketch this connection briefly in this section, although we will study the N Colony System in its own right thereafter. We take the definition of the diffusion from Chapter 7.1.2 of [DG2010].

Definition 1.18 (Fleming Viot Diffusion with two types)**.**
Let for $N \in \mathbb{N}$ *and given parameters*

$$c,\ s,\ m,\ d > 0 \tag{1.44}$$

the $\mathbb{R}^N$ *valued process*

$$y^N = (y_i^N(t)\ :\ i = 1, ..., N) \tag{1.45}$$

be the solution to the following system of stochastic differential equations:

$$\begin{aligned} y_i^N(0) &= 1\,, \\ dy_i^N(t) &= c\left(\overline{y}^N(t) - y_i^N(t)\right)dt - sy_i^N(t)(1 - y_i^N(t))dt - \frac{m}{N}y_i(t)dt \\ &\quad + \sqrt{d \cdot y_i^N(t)(1 - y_i^N(t))}dB_i(t) \\ &\qquad (i = 1, ..., N)\,. \end{aligned} \tag{1.46}$$

□ □ □□□

Here, the collection

$$\{B_i(\cdot)\,:\,i=1,\,...,\,N\} \tag{1.47}$$

denotes independent standard Brownian motions, and

$$\overline{y}^N(t)=\frac{1}{N}\sum_{j=1}^{N}y_j^N(t) \tag{1.48}$$

the average mass at time t.

Dawson and Greven show that this process is well-defined, and relate it to the process of the total particle number in the N Colony System via duality.

Theorem 1.19 (Duality)**.**
Let as before $\Pi^N(t)$ denote the total number of particles in the N Colony System. Then, the following holds:

1. *For fixed N, there exists a unique solution y^N to the martingale problem that is associated with the system of stochastic differential equation specified in Definition 1.18.*
2. *The following moment relation holds:*

$$\mathbb{E}\left[y_1^N(t)\right]=\mathbb{E}\left[\exp\left(-\frac{m}{N}\int_0^t \Pi^N(s)ds\right)\right]\,. \tag{1.49}$$

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Part I

The first time window

The goal of Part I is to examine the N Colony System in the first time window starting at time $t = 0$. In Chapter 2, the Collision Free System is examined, which is the potential limit candidate in the limit $N \to \infty$. Chapter 3 then shows convergence of the N Colony System towards the Collision Free System. This convergence holds when the paths starting at time $t = 0$ are considered and when a notion of convergence is used that ignores deviations that happen far to the right. For this, all processes are constructed on one single probability space; convergence is then shown to hold almost surely uniformly on compact time intervals.

2 The Collision Free System

The functional $K(t)$ as introduced in Definition 1.7 counts the number of occupied colonies in the Collision Free System. This chapter aims to prove the following exponential growth property of this quantity:

Theorem 2.1 (The exponential growth of the Collision Free System)**.**
Let s denote the individual birth rate of particles. Let $K(t)$ denote the number of occupied colonies in the Collision Free System ζ at time t. Let $\Psi(t)$ denote the statistics of the Collision Free System.

1. *Then, there exists a random variable W and a constant $0 < \alpha \leq s$ such that*

$$\lim_{t \to \infty} \frac{K(t)}{\exp(\alpha t)} = W \quad \text{a. s., in } L_1 \text{ and in mean square}. \tag{2.1}$$

   Moreover, $W > 0$ a. s. and

$$\mathbb{E}[W] < \infty, \ \mathbb{E}\left[W^2\right] < \infty. \tag{2.2}$$

2. *The normalized statistics converge towards some stable distribution, i. e. there exist a (nonrandom) distribution*

$$\Psi_\infty \in \mathcal{M}_{\leq 1}(\mathbb{N}) \tag{2.3}$$

   such that almost surely for all $j \in \mathbb{N}$

$$\frac{\Psi(t, j)}{K(t)} \to \Psi_\infty(j). \tag{2.4}$$

Remark 2.2. *The theorem states that $K(t) = (W + o(1))e^{\alpha t} \quad (t \to \infty)$. It will later be convenient to refer to this $o(1)$-deviation term in a concise way; we thus define*

$$w(t) = \frac{K(t)}{e^{\alpha t}} - W. \tag{2.5}$$

Hence, $w(t) \to 0$ almost surely and in L^1.

A reformulation of this result in terms of the hitting time $T_\cdot$ that will frequently be used in the subsequent chapters is given in Section 2.4. The proof of the Theorem is presented in Section 2.3; the arguments rely on certain bounds on the evolution of a single colony which are obtained in Section 2.2. The deeper reason why we need knowledge of what happens locally in order to understand the global behaviour is that $K(t)$ can be seen as a continuous time generalization of a Galton Watson branching process. Just as it is necessary in the Galton Watson analogue to know the offspring distribution of a single individual, it is in our context necessary to analyse the emigrants of a single colony. Because of this branching process analogue, the Collision Free System is also referred to as CMJ system, where CMJ is the abbreviation for Crump Mode Jagers. These authors investigated such age dependent branching processes.

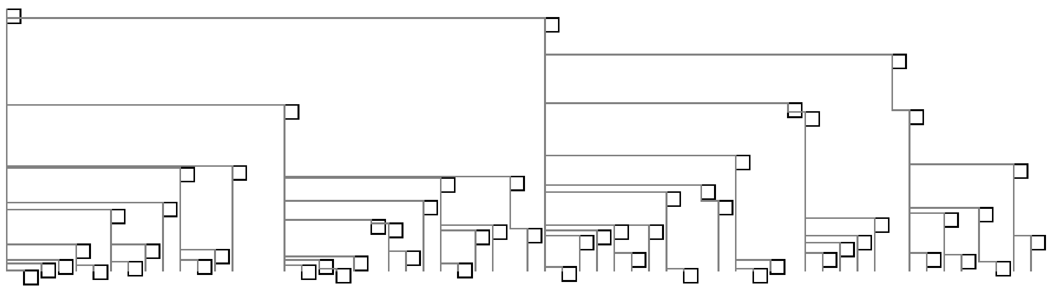


Figure 2.1: The migration tree. Time runs downwards; the lengths of horizontal lines have no special meaning. Each square resembles a colonized colony; the vertical lines indicate that the colony that they belong to are still inhabited. Horizontal lines show how the migrants move; the direction is always from the colony where the adjacent vertical line ends to the colony where the horizontal line ends. The parameters of the simulation are $s = 5$, $d = 0.5$, $c = 1$; the time axis ranges from 0 to 1.5.

Before going into the proof, it is first shown in Section 2.1 that the Collision Free System as well as the N Colony System are well-defined Markov processes. Finally, the Chapter is concluded by Section 2.5, where bounds on the number of single particle migrations (cf. Definition 1.2) are calculated.

2.1 Consolidation of the definitions

In order to consolidate the Definitions 1.1 and 1.3, we show first that the particle processes are well-defined. The property of interest is that with probability one no explosion occurs, i. e. that only a finite number of jumps occur in any finite time interval. This allows then to uniquely construct the process from a discrete time Markov Chain and exponentially distributed holding times. We take the definition of the explosion time from [JN1997], p. 69.

Definition 2.3 (Explosion of a pure jump type Markov process)**.**
For a given pure jump type Markov process $(Y(t))_{t\geq 0}$*, define the* jump times

$$J_0,\ J_1,\ J_2,\ \ldots \tag{2.6}$$

via

$$J_0 = 0,\ J_{n+1} = \inf\{t \geq J_n\ :\ Y(t) \neq Y(J_n)\}\,. \tag{2.7}$$

The explosion time

$$\zeta \in [0,\ \infty] \tag{2.8}$$

is defined as

$$\zeta = \limsup_{n\to\infty} J_n\,. \tag{2.9}$$

The process Y *is called* explosive *if and only if*

$$\mathbb{P}\,(\zeta < \infty) > 0\,. \tag{2.10}$$

Remark 2.4. *This definition shows also the usual decomposition of a jump process* Y *into its embedded jump chain (sometimes also called skeleton chain)*

$$(Y(J_n))_{n\geq 0} \tag{2.11}$$

and the waiting times

$$W_n = J_{n+1} - J_n \,. \tag{2.12}$$

It is shown e. g. in Theorem 2.8.2 of [JN1997] that a time-homogeneous nonexplosive process can uniquely be constructed from its embedded jump chain, which is itself Markovian, and the waiting times W_n, which are exponentially distributed with a parameter that depends only on the state $Y(J_n)$.

We show the non-explosion property only for the N Colony System, the proof for the Collision Free System being similar. The quick argument is that the total number of particles in the system is stochastically dominated by a pure linear birth process, which is known to be nonexplosive (cf. Theorem 2.5.2 of [JN1997], where it is shown that such a birth process is explosive if and only if the inverse of the birth rates is summable). In order to rule out the possibility that the additional events induced by the coalescence and migration mechanisms cause an explosion, we verify a standard criterion that is summarized in the Appendix in Theorem A.1 and that may be found in [DS2005] (Theorem 4.3.6) or [MC2004] (Theorem 0.5).

Proposition 2.5. *The N Colony System as specified in Definition 1.1 is nonexplosive.*

Proof. Define

$$F_M = \{1,\ \dots,\ M\}^N \tag{2.13}$$

and for $(v_1,\ \dots,\ v_N) \in \mathbb{S}^N$

$$u(v_1,\ \dots,\ v_N) = \sum_{k=1}^{N} v_k \,. \tag{2.14}$$

Additionally, we define the operators $T_k^+,\ T_k^-$ on $w = (w_1,\ \dots,\ w_N) \in \mathbb{S}^N$ as follows:

$$\begin{aligned} T_k^+ w &= (w_1,\ \dots,\ w_{k-1},\ w_k + 1,\ w_{k+1},\ w_N)\,, \\ T_k^- w &= (w_1,\ \dots,\ w_{k-1},\ w_k - 1,\ w_{k+1},\ w_N)\,. \end{aligned} \tag{2.15}$$

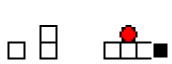
Let $P_{v,w}$ be the probability to leave state v in favour of state w. Noting that migration leads to an increase in one colony and to a decrease in another, we may write for fixed $v \in \mathbb{S}^N$

$$\begin{aligned} \sum_{w\in\mathbb{S}^N} P_{v,w} u(w) &= \sum_{k=1}^{N} \Big[P_{v,T_k^+ v} u(T_k^+ v) + P_{v,T_k^- v} u(T_k^- v) 1_{\{v_k > 1\}} \Big] \\ &\quad + \sum_{k,\,l} P_{v,T_k^+ T_l^- v} u(T_k^+ T_l^- v) 1_{\{v_l \geq 1\}} \\ &= \sum_{k=1}^{N} \Big[P_{v,T_k^+ v} (u(v)+1) + P_{v,T_k^- v} (u(v)-1) 1_{\{v_k > 1\}} \Big] \\ &\quad + u(v) \sum_{k,\,l} P_{v,T_k^+ T_l^- v} 1_{\{v_l \geq 1\}} \\ &= u(v) + \sum_{k=1}^{N} \Big[P_{v,T_k^+ v} - P_{v,T_k^- v} 1_{\{v_k > 1\}} \Big] \,. \end{aligned} \tag{2.16}$$

Neglecting the decreases due to deaths and using the relationships between transition rates, holding time parameters and jump probabilities that are summarized in the equations (A.2), we can rewrite this as follows:

$$\sum_{w\in\mathbb{S}^N} P_{v,w} u(w) \leq u(v) + \frac{1}{R_v} \sum_{k=1}^{N} s v_k = u(v)(1 + \frac{s}{R_v})\,. \tag{2.17}$$

Here, R_v denotes the total rate at which state v is left. We can now apply Theorem A.1. □



2.2 A single colony in equilibrium

In order to find bounds on the population on one single colony (and thus, in a later step, on the number of emigrations), we study first such a colony in equilibrium. It will then be argued below in Proposition 2.8 that a given population of finite age can always stochastically be dominated by its equilibrium.

We neglect the possibility that the last inhabitant of the fixed colony emigrates; otherwise, we would not obtain an equilibrium. This is justified by the fact that, in the Collision Free System, a single particle migration causes only a relabelling of colonies.

Proposition 2.6. *Let s, $d > 0$ and $c \geq 0$ be given constants. Consider the $\mathbb{N}$ valued birth and death process $(z(t))_{t\geq 0}$ with initial value 1 and*

$$\text{birth rate } ns \text{ and death rate } cn\mathbb{1}_{\{n\geq 2\}} + \tfrac{d}{2}n(n-1)\,, \tag{2.18}$$

when there are n particles present. Then, there exists a unique stationary distribution $(\pi(j) : j \in \mathbb{N})$ given by

$$\pi(j) = \pi(1)\prod_{k=2}^{j} \frac{s(k-1)}{ck + \frac{d}{2}k(k-1)}\,, \tag{2.19}$$

where $\pi(1)$ is chosen such that the vector $(\pi(j))$ sums to 1. Furthermore, for all $m \in \mathbb{N}$,

$$\sum_{j\geq 1} j^m \pi(j) < \infty\,. \tag{2.20}$$

Proof. Kelly shows in Chapter 1.3 of [FK1979] that a birth and death process with values in $\mathbb{N}$ and birth rate b_r and death rate d_r, when there are r individuals present, has an equilibrium distribution $(p(j))_j$ if and only if the detailed balance equation is satisfied, that is

$$p(j) = p(1)\prod_{r=2}^{j} \frac{b_{r-1}}{d_r} \tag{2.21}$$

such that this is summable. From this, the formula (2.19) follows. The sum over (2.19) as well as all its moments are finite; this can be seen by the following comparison with elements of an exponential series:

$$\sum_{j\geq 1} j^m \prod_{k=2}^{j} \frac{s(k-1)}{ck + \frac{d}{2}k(k-1)} \leq \sum_{j\geq 1} j^m \frac{1}{j!}\left(\frac{2s}{d}\right)^{j-1} < \infty\,. \tag{2.22}$$

□

Remark 2.7. *In Chapter 6, the equilibrium of the N Colony System is calculated. In an additional heuristic argument, the present single colony birth and death process will be equipped with a constant immigration stream at fixed deterministic rate $\iota > 0$. This Poisson immigration stream is a simplification of the true time inhomogeneous immigration that depends on the occupancy numbers in the other colonies. The equilibrium for such a colony is calculated in Proposition 6.7. It is shown in Remark 6.8 that, for the correct value of ι, the obtained distribution indeed equals the marginal law of the equilibrium of the N Colony System at one single colony.*

We turn to the evolution in time of the colony. The equilibrium measure will be used below to bound the time marginal for fixed t. This is justified by the following coupling argument. The coupling idea is the same that is used in Chapter 3 to compare ζ and ζ^N and is taken from Chapter 7 of [DG2010].

Proposition 2.8. *For $(z(t))_{t\geq 0}$ as in Proposition 2.6 and times u, v with $0 \leq u < v$, the following stochastic orderings hold:*

$$z(0) \leq_{st} z(u) \leq_{st} z(v) \leq_{st} z(\infty) \ . \tag{2.23}$$

Here, $z(\infty)$ denotes the population in equilibrium.

Proof. The first stochastic ordering is trivial. We first show the third. Consider a population

$$(z_\infty(t))_{t\geq 0} \tag{2.24}$$

in equilibrium (i. e. a population that starts with the equilibrium distribution); this implies

$$\mathcal{L}\left(z_\infty(0)\right) = \mathcal{L}\left(z_\infty(t)\right) \text{ for all } t \geq 0 \ . \tag{2.25}$$

Now we introduce colours and couple thereby the states of the original population $z(\cdot)$ at different times.

- Initially, mark all particles of $z_\infty(0)$ white. Give to one randomly chosen particle a special colour, say green. Let the offspring adopt the colour of the parent particle.
- Arrange that no coalescence event of equal colours changes the colours, i. e. a pair of white particles coalesces to a white particle and a pair of green particles coalesces to a green particle.
- Finally, a pair of a green and a white particle shall coalesce to a green particle. This has the effect that the green population is not affected by the white population that it is embedded in.

By construction, the evolution of the green population is a version of

$$(z(t))_{t\geq 0} \ , \tag{2.26}$$

while the evolution of the whole population, ignoring colours, equals

$$(z_\infty(t))_{t\geq 0} \ . \tag{2.27}$$

In this realization, i. e. when $z(v)$ counts the green particles, we have

$$z(v) \leq z_\infty(v) \tag{2.28}$$

almost surely. This together with (2.25) yields the claim.

The second stochastic ordering follows similarly: Define

$$r = v - u \, , \tag{2.29}$$

realize the system $(z(t))_{t\geq 0}$ and pick at time r one particle at random and colour it green. Tracking the green offspring as before, we obtain at time v two populations: A green population of age $v - (v - u) = u$ and the total population which is of age v. By construction, the older population cannot be smaller than the younger population. □

2.3 Proof of Theorem 2.1

The Collision Free System can be interpreted as a birth process, where birth means colonization of a previously unoccupied colony. Keeping track of colonizations establishes a tree-like structure, and we will apply a convergence result about general branching processes taken from Nerman's 1981 paper ON THE CONVERGENCE OF SUPERCRITICAL GENERAL (C-M-J) BRANCHING PROCESSES (cf. [ON1981]).

We first specify the point process ξ that counts the emigrants from a single colony. This process can be seen as the age dependent analogue of the offspring distribution of a single individual in the Galton Watson process. It would be convenient for the analysis if ξ were a doubly stochastic Poisson process, i. e. a standard Poisson process with random but independent rate. This is false because at any event of ξ the population decreases by one, but ignoring this decrease gives a process of the desired nature. This can be used to bound the emigration stream from above. The following is a technical argument that will be needed below in the actual proof.

Lemma 2.9. *Let the constants s, $d > 0$ and $\tilde{c} \geq 0$ be given. Consider like in Proposition 2.6 a birth and death process $(z_{\tilde{c}}(t))_{t\geq 0}$ with initial value 1 and*

$$\text{birth rate } ns \text{ and death rate } \tilde{c}n1_{\{n\geq 2\}} + \frac{d}{2}n(n-1)\,, \tag{2.30}$$

when there are n particles present. An emigration is said to happen whenever a particle dies due to the linear death rate $\tilde{c}$. Let the random measure ξ be defined via

$$\xi(I) = \text{number of emigrations in time interval } I\,, \tag{2.31}$$

where $I \subset \mathbb{R}_+$. Let

$$\mu(I) = \mathbb{E}\left[\xi(I)\right] \tag{2.32}$$

denote the expected number of emigrations.

1. *For any non-negative and Borel-measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$,*

$$\mathbb{E}\left[\int_{\mathbb{R}_+} f(s)\xi(ds)\right] = \int_{\mathbb{R}_+} f(s)\mu(ds) \leq \tilde{c}\mathbb{E}\left[z_0(\infty)\right]\int_{\mathbb{R}_+} f(s)ds\,, \tag{2.33}$$

where $z_0(\infty)$ denotes the equilibrium state of the birth and death process with emigration rate $\tilde{c} = 0$.

2. *The second moment can be estimated similarly:*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+} f(s)\xi(ds)\right)^2\right] \leq \left(\tilde{c}^2\mathbb{E}\left[(z_0(\infty))^2\right] + \tilde{c}\mathbb{E}\left[z_0(\infty)\right]\right) \cdot \left(\sum_{m\geq 0} \sup_{s\in[m,\, m+1)} f(s)\right)^2 . \tag{2.34}$$

Proof. Using the usual approximation scheme, it is sufficient to show the first identity of (2.33) for indicator functions. For $A = (a,\ b]$ and arbitrary $a,\ b$ in $\mathbb{R}_+$, we have

$$\mathbb{E}\left[1_A(t)\xi(dt)\right] = \mathbb{E}\left[\xi(A)\right] = \mu(A)\,, \tag{2.35}$$

and this shows the identity for arbitrary A using Carathéodory's extension theorem.

In order to see the inequality in (2.33), we show that the integral with respect to μ is smaller than the corresponding integral with respect to a certain measure μ_0, which finally leads to the right hand side of (2.33). Modify the birth and death process such that emigrations leave the population size unaltered; let the measure counting emigrations be denoted by ξ_0. The corresponding intensity measure μ_0 can be constructed as follows: Take a standard Poisson process $P(\cdot)$ and define

$$\mu_0([0,\, t)) = \mathbb{E}\left[P\left(\tilde{c}\int_{[0,\, t)} z_0(s)1_{\{z_0(s)\geq 2\}}ds\right)\right]\,, \tag{2.36}$$

where z_0 is a birth and death process with emigration parameter $\tilde{c} = 0$. The claim

$$\mu([a,\, b)) \leq \mu_0([a,\, b)) \tag{2.37}$$

follows via coupling: In the right system, there are more particles, and the emigration rate is higher; the increase is thus at least as high as in the left system. In particular, using the usual calculus for doubly stochastic Poisson processes and Proposition 2.8, we obtain

$$\mu_0([0,\, t)) = \tilde{c} \int_{[0,\, t)} \mathbb{E}\left[z_0(s) 1_{\{z_0(s) \geq 2\}}\right] ds \leq \tilde{c} t \mathbb{E}\left[z_0(\infty)\right] . \tag{2.38}$$

Hence,

$$\int_{\mathbb{R}_+} f(s)\mu(ds) \leq \int_{\mathbb{R}_+} f(s)\mu_0(ds) \leq \tilde{c}\mathbb{E}\left[z_0(\infty)\right] \int_{\mathbb{R}_+} f(s) ds \, . \tag{2.39}$$

For the claim (2.34), abbreviate

$$f^+(m) = \sup_{s \in [m,\, m+1)} f(s) \, , \tag{2.40}$$

and use Cauchy-Schwarz in order to obtain

$$\begin{aligned} \mathbb{E}\left[\left(\int_{\mathbb{R}_+} f(s)\xi(ds)\right)^2\right] &\leq \mathbb{E}\left[\left(\sum_{m \geq 0} \xi([m,\, m+1)) f^+(m)\right)^2\right] \\ &\leq \left(\sum_{m \geq 0} f^+(m) \sqrt{\mathbb{E}\left[(\xi[m,\, m+1))^2\right]}\right)^2 . \end{aligned} \tag{2.41}$$

Using Jensen's Inequality on the probability space

$$([m,\, m+1),\, \mathcal{B}[m,\, m+1),\, \lambda_{|[m,\, m+1)}) \tag{2.42}$$

and, as before, Fubini and Proposition 2.8, we obtain

$$\begin{aligned} \mathbb{E}\left[(\xi[m,\, m+1))^2\right] &= \mathbb{E}\left[\left(c \int_m^{m+1} z_0(s) ds\right)^2 + \left(c \int_m^{m+1} z_0(s) ds\right)\right] \\ &\leq \mathbb{E}\left[c^2 \int_m^{m+1} (z_0(s))^2 ds + c \int_m^{m+1} z_0(s) ds\right] \\ &\leq c^2 \mathbb{E}\left[(z_0(\infty))^2\right] + c\mathbb{E}\left[z_0(\infty)\right] . \end{aligned} \tag{2.43}$$

□

We are now ready to prove Theorem 2.1. We repeat the main argument: Due to the absence of collisions, colonies evolve independently once they are born. Each colony evolves in time according to a certain distribution on the path space; viewed from the outside, this corresponds to a random point process, where each event represents an emigrating particle founding a new colony.

In other words, all colonies give birth to new colonies according to identically distributed independent point processes that only depend on the age of the particular parent colony. This is the definition of a Crump-Mode-Jagers (CMJ) Branching Process, as discussed in [ON1981]. From this point of view, $K(t)$ denotes the size of the (colony) population at time t.

Proof of the first assertion of Theorem 2.1. We apply the results obtained by Nerman in his article [ON1981]. These are summarized in Appendix A.2. Let ξ and μ be as in Lemma 2.9. The bounds on α can immediately seen by a comparison with a simple birth process of rate s.

Now we check the conditions of Theorem A.4. Clearly, μ is of non-lattice type. Also, there exists some $\alpha \in (0,\ \infty)$ such that

$$\begin{aligned} \int_0^\infty e^{-\alpha t}\mu(dt) &= 1\,, \\ \int_0^\infty te^{-\alpha t}\mu(dt) &< \infty\,, \end{aligned} \tag{2.44}$$

which follows immediately from Lemma 2.9. Finally, in order to see

$$\mathbb{E}\left[Z\,[\log Z]^+\right] < \infty\,, \tag{2.45}$$

where

$$Z = \int_0^\infty e^{-\alpha t}\xi(dt)\,, \tag{2.46}$$

it is sufficient to show

$$\mathbb{E}\left[Z^2\right] < \infty\,. \tag{2.47}$$

By virtue of Lemma 2.9, this is implied by

$$\sum_{m\geq 0} e^{-\alpha m} < \infty\,. \tag{2.48}$$

This implies almost sure and L^1 convergence. Existence of the second moment and mean square convergence follow then also from the estimate (2.47). Here, we apply Theorem 1 of [GD1974], which is summarized in Theorem A.6 in Appendix A.2. □

Remark 2.10. *The calculations in Lemma 2.9 show that the m^{th} moment of the variable Z as defined in equation* (2.46) *is finite if and only if the m^{th} moment of the variable $z_0(\infty)$ is finite (since the m^{th} moment of a Poisson process of rate λ is a polynomial in λ of degree m). Proposition 2.6 ensures that all moments of $z_0(\infty)$ are finite; Theorem 1 of [BD1975], which is summarized in Theorem A.5 in Appendix A.2, is thus applicable and yields that also the m^{th} moment of W is finite, for any $m \in \mathbb{N}$. We will not make use of these higher moments, so we exclude a formal proof.*

We now finish the proof.

Proof of the second assertion of Theorem 2.1. We apply Corollary 6.4 of [ON1981] which is summarized in Theorem A.7 in Appendix A.2. It follows again from Lemma 2.9 that

$$\int_0^\infty e^{-\beta s}\mu(ds) < \infty \tag{2.49}$$

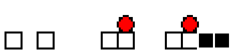
for any $\beta \in (0,\ \alpha)$. The cited result then yields almost sure convergence of age ratios, i. e.

$$\lim_{t\to\infty} \frac{K^T(t)}{K(t)} = 1 - e^{-\alpha T}\,, \tag{2.50}$$

where $K^T(t)$ denotes the number of colonies at time t that have been invaded at some time s, where $s \geq t - T$. Since the distribution of particles on a given collection of colonies depend only on their respective age, there exists a distribution Ψ_∞ such that

$$\mathcal{L}\left(\frac{\Psi(t)}{K(t)}\right) \Rightarrow \delta_{\Psi_\infty}\,. \tag{2.51}$$

The limiting distribution is concentrated on one point because of a law of large numbers on many parallel colonies of the same age. This together with the stochastic monotonicity of a fixed colony $z(t)$ (cf. Proposition 2.8) yields almost sure convergence for each component. □



2.4 Consequences for the hitting times

Recall that $T_\cdot$ is the hitting time of the functional $t \mapsto K(t)$ at level $\cdot$ (cf. Definition 1.8).

Corollary 2.11. *The Collision Free System grows asymptotically deterministically: For any non-negative functions f, g with $f(N) \to \infty$ for $N \to \infty$ and $f \le g$, the following convergence holds almost surely:*

$$\begin{aligned} \lim_{N\to\infty} \left[T_{f(N)} - \frac{1}{\alpha} \log f(N) \right] &= \frac{1}{\alpha} \log W\,, \\ \lim_{N\to\infty} \left[T_{f(N)} - T_{g(N)} - \frac{1}{\alpha} \log \frac{f(N)}{g(N)} \right] &= 0\,. \end{aligned} \tag{2.52}$$

In particular, the difference of the hitting times $T_{g(N)} - T_{f(N)}$ is itself asymptotically deterministic.

Proof. By the non-explosion property, $T_{f(N)} \to \infty$ almost surely for $N \to \infty$. The claim follows thus immediately from Theorem 2.1 because of the representation

$$T_{f(N)} = \frac{1}{\alpha} \log \left(\frac{f(N)}{W + w(T_{f(N)})} \right) \tag{2.53}$$

that is obtained by inversion of the growth rule (2.1). □

Remark 2.12. *A similar statement is possible for the N Colony System. Here, the branching process structure is disturbed by circular migration, but we can still give bounds by comparison with CMJ processes. This assigns to the hitting time $T^N_{\epsilon N}$ an asymptotically deterministic but slowly growing corridor to live in. The details are as follows:*

Assume that one can realize the N Colony System ζ^N and the Collision Free System ζ on one probability space such that the trajectories are equal as long as no migration occurs. Assume that at each migration a coin is tossed that decides if the particle collides or not. Assume further that one can ensure by coupling

$$K^N(t) \le K(t)\,. \tag{2.54}$$

Up to time $T^N_{\epsilon N}$, the collision probability is bounded by ϵ; we can thus introduce a system $K_\epsilon(t)$ that is obtained by replacing the varying collision probability in the coin tossing by the constant ϵ. Additionally, each colliding particle is not placed in a randomly chosen occupied colony but completely removed from the system. The independently thinned system $K_\epsilon(t)$ is again a branching process; the almost sure ordering

$$K_\epsilon(t) \le K^N(t) \le K(t) \tag{2.55}$$

implies

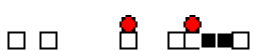
$$\frac{1}{\alpha} \log \frac{\epsilon N}{W} + o(1) \le T^N_{\epsilon N} \le \frac{1}{\tilde\alpha} \log \frac{\epsilon N}{\tilde W} + o(1) \tag{2.56}$$

with α, W as in Theorem 2.1 and a number $\tilde\alpha \le \alpha$ and a nondegenerated random variable $\tilde W$.

In Chapter 3, it will be shown that it is possible to find a coupling that matches these requirements. This idea will not be picked up again because Corollary 4.14 will show, using different techniques, that there exists a compact set K such that

$$T^N_{\epsilon N} - \frac{1}{\alpha} \log N \in K \tag{2.57}$$

for all large N.



2.5 Bounds on single particle migrations

In Chapter 4 it will be shown that, for fixed small $\epsilon > 0$, up to time $T_{\epsilon N}$ the trajectories $t \mapsto K^N(t)$ and $t \mapsto K(t)$ do not differ much. As a preparation towards this result, we show that the number of migrations that leave $K(t)$ unchanged can not get blatantly large. Recall (cf. Definition 1.2) that these events are called single particle migrations because they are caused by migrating particles that previously occupied a colony by itself.

A bound on the number of single particle migrations is important because at each such migration the particle has the chance to collide, while $K(t)$ does not come any closer to $\lceil \epsilon N \rceil$; a large number of single particle migrations thus threatens to tear apart the trajectories of $K^N(t)$ and $K(t)$, when these objects are suitably defined on a common probability space.

Definition 2.13 (number of single particle migrations)**.**
Let the $\mathbb{N}$*-valued random variable*

$$s(i) \tag{2.58}$$

denote the number of migrations in the Collision Free System between T_i *and* T_{i+1}*. Equivalently,* $s(i) - 1$ *is the number of single particle migrations after the* $(i-1)^{th}$ *increase of* $K(t)$*.*

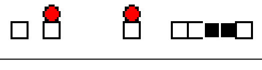
The following Proposition gives two bounds on the single particle migrations: The fourth point shows that the greatest $s(i)$ between 1 and $\lceil \epsilon N \rceil$ is of order $\log N$, while the fifth shows that the accumulated single particle migrations up to level $\lceil \epsilon N \rceil$ is of order N. The latter bound will turn out to be the more useful one.

Proposition 2.14. *Let*

$$p_0 \equiv p_0(s,\ d,\ c) \tag{2.59}$$

denote the probability that the first migrant of the first colony is not single.

1. *For* p_0*, there are the following bounds:*
$$\left(\frac{s}{c+s}\right)\left(\frac{2c}{2c+2s+d}\right) \le p_0 \le \frac{s}{c+s}\,. \tag{2.60}$$
2. *The quantity* $s(1)$ *is geometrically distributed with parameter* p_0*.*
3. *There exists an i. i. d. sequence* $\{s^{(j)}(1) : j \in \mathbb{N}\}$ *of copies of* $s(1)$ *such that for all* $i \in \mathbb{N}$
$$s(i) \le s^{(i)}(1) \tag{2.61}$$
holds almost surely. Due to coupling, the $s^{(j)}(1)$ *are not independent of* $(\Psi(t))_{t \ge 0}$*.*
4. *There exists only a finite number of values of* N *such that, up to time* $T_{\epsilon N}$*, more than* $3p_0^{-1} \log N$ *single particle migrations occur between two increases of* $K(t)$*, i. e.*
$$\limsup_{N\to\infty} \sup_{1 \le i \le \lceil \epsilon N \rceil - 1} \frac{s(i)}{3p_0^{-1} \log N} \le 1\,. \tag{2.62}$$
5. *There exists only a finite number of values of* M *such that the number of accumulated single particle migrations up to level* M *is greater than* $p_0^{-1}(p_0+1) \cdot M$*, i. e.*
$$\limsup_{M\to\infty} \left[\sum_{i=1}^{M} s(i) - M\left(\frac{1}{p_0} + 1\right)\right] \le 0\,. \tag{2.63}$$
6. *Let* $\tau(M)$ *denote the time of the* M^{th} *migration. Then, for large* M*,*
$$T_{\frac{p_0}{1+p_0}M} \le \tau(M) \le T_{M+1}\,. \tag{2.64}$$



We first prove the first three claims, namely that the sequence $\{s(i)\}$ can almost surely be bounded by an i. i. d. sequence $\{s^{(i)}(1)\}$ of geometrically distributed random variables with nondegenerated parameter. All later calculations can then be simplified by replacing the dependent sequence $\{s(i)\}$ by its i. i. d. backbone $\{s^{(i)}(1)\}$. In particular, the fifth assertion is a direct consequence of the strong law of large numbers; the bound $M(p_0^{-1}+1)$ could thus be replaced by $M(p_0^{-1}+\eta)$ for any prefixed $\eta > 0$.

Proof of the first three assertions. For the first assertion, note that $1 - p_0$ can be bounded from below by the probability that the very first event is a single particle migration; i. e.

$$1 - p_0 \geq \frac{c}{c+s}\,. \tag{2.65}$$

Similarly, p_0 is bounded from below by the probability that the first two events are a birth succeeded by a migration:

$$p_0 \geq \left(\frac{s}{c+s}\right)\left(\frac{2c}{2c+2s+d}\right)\,. \tag{2.66}$$

For the second assertion, note that after each single particle migration, the colony falls back into its initial state (albeit with another colony index), and $s(1)$ is therefore geometrically distributed with parameter p_0.

The third claim holds, because $s(1)$ is caused by a colony that initially carries only one particle; if now i colonies act in parallel and independently, the worst possible situation is that each colony carries precisely one particle. A superposition of these i independent migration streams behaves exactly like a single one, when the only quantity of interest is the number of single particle migrations in opposition to "true" migrations. We formalize this via coupling:

- Assume $K_t = i$. After any migration step (be it "single" or "true"), colour precisely one particle per colony with a special colour, say green; colour their offspring the same way. Arrange that the coloured population is not influenced by any coalescence event where an uncoloured particle is involved, using coalescence rules similar to those seen in the proof of Proposition 2.8.

- The crucial fact is that the remaining uncoloured population may only produce true migrations and no single particle migrations. Hence, in order to find an upper bound on $s(i)$, we may completely ignore this population.

- So long as $K_t = i$, the offspring evolution of green particles is independent between green migration events: After a green single particle migration, all colours are erased and precisely i particles are chosen at random to be coloured again. By convention, these are not influenced by their surroundings, and thus do not make use of any information that was gathered earlier. This renewal structure implies that the first appearance of a "true" green migration amongst the single particle migrations is geometrically distributed.

- When a green migration occurs, it is equally likely to be from any of the i green populations on the respective colonies; any of these populations evolve in law like $\zeta_1(t)$ starting from $t = 0$. This identifies the parameter of the geometric distribution as p_0.

□

Having revealed the important structural property that the sequence $\{s(i)\}$ consists mainly of i. i. d. random variables, the proof of the remaining assertions is simple.

Proof of the remaining assertions. For the fourth assertion, calculate

$$\begin{aligned}\mathbb{P}\left(\max_{i=1,\,\ldots,\,\lceil\epsilon N\rceil-1} s(i) > 3p_0^{-1}\log N\right) &\leq \sum_{i=1}^{\lceil\epsilon N\rceil-1}\mathbb{P}\left(\exp\left(p_0 s^{(i)}(1)\right) > N^3\right)\\ &\leq \frac{(\lceil\epsilon N\rceil-1)}{N^3}\mathbb{E}\left[\exp(p_0 s(1))\right], \end{aligned} \tag{2.67}$$

and this is summable provided that the exponential moment of $s(1)$ is finite. The moment generating function of a geometrically distributed random variable with parameter p_0 exists at point t if and only if

$$e^t(1-p_0) < 1\,, \quad \text{i. e. if} \quad t < -\log(1-p_0)\,. \tag{2.68}$$

This again is certainly the case for $t \leq p_0$, as can be seen using the Taylor expansion of the logarithm.

The fifth assertion follows from the strong law of large numbers. For an explicit calculation, consider some $t > 0,\ a > 1$ and calculate

$$\mathbb{P}\left(\sum_{i=1}^{M} s(i) > aM\right) \leq \exp(-atM)\prod_{i=1}^{M}\mathbb{E}\left[\exp\left(ts^{(i)}(1)\right)\right] = \left(\frac{p_0 e^{-t(a-1)}}{1-(1-p_0)e^t}\right)^M. \tag{2.69}$$

For fixed a, the expression is minimal if

$$t = \log\left(\frac{a-1}{a(1-p_0)}\right), \tag{2.70}$$

and this leads to

$$\mathbb{P}\left(\sum_{i=1}^{M} s(i) > aM\right) \leq \left(ap_0\left[\frac{1-p_0}{1-\frac{1}{a}}\right]^{(a-1)}\right)^M. \tag{2.71}$$

In particular, when $a = (1+p_0)p_0^{-1}$, one obtains the bound

$$\mathbb{P}\left(\sum_{i=1}^{M} s(i) > aM\right) \leq \left((1+p_0)\left[1-p_0^2\right]^{\frac{1}{p_0}}\right)^M, \tag{2.72}$$

and the expression in brackets is smaller than 1 for any $p_0 \in (0,\ 1)$, which implies that (2.72) is summable.

Finally, the last assertion follows from the fifth, because if K is large enough to ensure

$$\sum_{i=1}^{K} s(i) \leq \frac{1+p_0}{p_0}K\,, \tag{2.73}$$

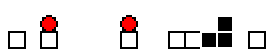
then

$$\sum_{i=1}^{K} s(i) \leq M \tag{2.74}$$

is satisfied when

$$K \leq \left(\frac{1+p_0}{p_0}\right)^{-1} M\,. \tag{2.75}$$

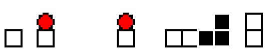
Inequality (2.74) then ensures that no more than M migrations are necessary to reach level $K+1$, i. e. $T_{K+1} \leq \tau(M)$. The second inequality is trivial because there are at least M migrations necessary to reach level $M+1$. $\square$



Remark 2.15. *At first sight, it looks as if the approximation of $s(i)$ by $s(1)$ is a vast overestimation; one might suspect that a law of large numbers eradicates most of the fluctuations when there are many colonies present. But this is not the case: for $t \to \infty$, it is known that the distribution of the normalized occupancy numbers $K^{-1}(t)\Psi(t)$ converge to some stable size distribution Ψ_∞. For large i, $s(i)$ is thus close in distribution to a geometrically distributed random variable $s(\infty)$ with parameter*

$$1 - \frac{\Psi_\infty(1)}{\sum_{k\geq 1} k\Psi_\infty(k)}, \tag{2.76}$$

which is the proportion of particles that are not single. Hence, the parameter improves, but the distribution itself does not degenerate. (The situation would change when considering the time *between increases of $K(t)$ which we expect to be asymptotically deterministic.)*



3 The Multicolour Particle System

In this chapter, both the N Colony System ζ^N and the Collision Free System ζ will be constructed on one probability space. The goal is to approximate the N Colony System by the Collision Free System, at least in an early time window.

The idea is to let ζ^N follow the trajectories of ζ as long as no migration occurs. At any such migration event, a coin is tossed that decides whether the corresponding particle of ζ^N would collide; at any such collision, the affected particle together with its future offspring leave the coupling and evolve separately. This construction allows to show that initially both systems are close, almost surely and thus in law. This is proven in Theorem 3.6 below.

3.1 The multicolour coupling introduced by Dawson and Greven

In Step 2 of Section 7.4.6 of [DG2010], a multicolour particle system is constructed in order to incorporate both the dynamics of the N Colony System and of the Collision Free System; colours are assigned to particles according to their affiliation to the different systems. We will depart from this construction below and use a refined coupling instead that makes it necessary to introduce a rather long list of coupling rules. In order to communicate the basic idea as well as to point out the similarities and the differences to the coupling of Dawson and Greven, we quickly quote the description from [DG2010]:

> The multicolour comparison system has black, white and red particles. It has white and red particles located at the sites $\{1, ..., N\}$ and black particles located at a site in $\mathbb{N}$ where $\{1, ..., N\}$ and $\mathbb{N}$ are disjoint finite and countable sets respectively. [...] The initial state is given by having only white particles.

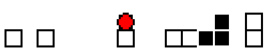
Further below, the result of the (yet to be introduced) evolution rules is summarized:

> Hence the key observation about the new system is that:
>
> - The number of occupied sites in the union of the black and white particles follows the dynamics of the process without collisions, i. e. the number of sites they occupy is a version of $K(t)$.
> - The number of occupied sites in the union of the white and red particles follows the exact [N Colony] dynamics, i. e. the number of sites they occupy is producing a version of $K^N(t)$.

Another consequence is that the difference between both particle numbers is given by the difference of the black and red particles; this is used below in Chapter 4 to quantify the deviation of both systems.

The main idea is to begin with white particles and to produce at each collision a pair of red and black particles; the red particle resembles the collided particle and the black the very particle if it had not collided. The red particles evolve on top of white particles; since the white particles shall, together with the black particles, form the Collision Free System, the red particles must not affect these white particles. The following evolution rules of Dawson and Greven show that this is possible such that, at the same time, the union of red and white particles gives the N Colony System:

> This runs as follows:
>
> - White particles at a site follow the same local dynamics as the dual particle system as far as birth (of white particles) and coalescence goes, changes occur for migration.
>   Let k denote the number of sites having currently at least one white or red



particle. Each migrating white particle moves with probability $1 - \frac{k}{N}$ to a new site in $\{1, ..., N\}$ which prior to this event did not contain any white or red particles and with probability $\frac{k}{N}$ changes to a black particle now located at a new unoccupied site in $\mathbb{N}$ and at the same time also a red particle is produced at an occupied site in $\{1, ..., N\}$ chosen at random among the k occupied sites. [...]

- Red particles have the same dynamics as the [N Colony] particle system on $\{1, ..., N\}$ (newborn particles are also red) and in addition when a red and white at the same site coalesce the outcome is always white.
- Black particles follow the same dynamics as the white except that migrating black particles move on $\mathbb{N}$ and always go to a new, so far unoccupied site in $\mathbb{N}$.

The complicated part is the second point where the colour of the outcome of a coalescence event is specified; recall that we have used similar coupling rules in the proof of Proposition 2.8. The effect of this rule is that, despite the quadratic death rate, the white population is not influenced by the presence of red particles.

Below, we shall depart from this original multicolour particle system as defined in [DG2010] for the following reasons:

- It is shown in [DG2010] that
$$K^N(t) \leq K(t) \tag{3.1}$$
holds stochastically. However, in the original multicolour particle system, one can easily construct situations where this does not hold almost surely: Consider a red population at one location that grows quickly while its black sister colony remains small. If the red particles spread out to unoccupied colonies, this may lead to
$$K^N(t) > K(t)\,. \tag{3.2}$$
We will strengthen the coupling below in order to ensure that (3.1) holds almost surely, essentially by coupling birth and migration events of the red and black twin particles.
- Furthermore, the multicolour particle system as defined in [DG2010] is formulated for previously fixed N. When comparing different values of N, also different realizations of the Collision Free System are compared. In order to be able to make use of the properties of the very same realisation of ζ uniformly in N, the multicolour model is refined below. All N Colony Systems will be constructed on the same probability space, starting from a single realisation of ζ.

In a similar spirit, Dawson and Greven introduce around Lemma 7.25 of [DG2010] additional colours that allow to gain more information about the difference of both systems. This track will not be followed here.

On the downside (besides a more complicate formulation), the modifications that the coupling needs to undergo make the following changes in the formulation necessary:

- The distinction between the colony space $\{1, ..., N\}$ where white and red particles live and the colony space $\mathbb{N}$ for black particles will be removed. Instead, particles of all colours live on $\mathbb{N}$ with the restriction that the number of colonies inhabited either by red or white particles does not exceed N.
- It is of no importance in [DG2010] whether single black particles are allowed to migrate because this only causes a relabelling of colonies. In contrast, we must allow such migrations here; otherwise, if a fixed realisation of ζ provides the white and black particles for all N as sketched above, possible collision opportunities would go unnoticed.

3.2 An uniform Multicolour Particle System

We use the Collision Free System ζ as defined in Definition 1.3 as starting point. For a given realisation of ζ, we now write down the rules how for fixed N the N Colony System ζ^N shall evolve on top of ζ. This yields a label equivalent modification of ζ^N (cf. Definition 1.12).

The description remains verbal; a possible Polish state space is sketched below in Section (3.4). In order to avoid too much overlap, we use the opportunity to stress there the *annealed* point of view (the Collision Free System and the N Colony System evolve simultaneously in time), contrary to the *quenched* point of view that is taken here (the N Colony System is constructed given a realisation of ζ).

Definition 3.1 (The Multicolour Particle System, quenched point of view)**.** *Let $\mathbb{S}^{col}$ be an enrichment of the state space $\mathbb{S}$ that allows to assign to each particle one of the three colours white, black, and red. We define on one probability space a collection of $\mathbb{S}^{col}$-valued processes*

$$\zeta^{col}, \ \{\zeta^{col,N} \ : \ N \in \mathbb{N}\} \tag{3.3}$$

and refer to this collection as the Multicolour Particle System*. The processes $\zeta^{col,N}$ are defined based on a realisation of ζ^{col} such that the information of ζ^{col} lives on in each $\zeta^{col,N}$, i. e. there exists some functional F (counting black and white colonies) such that*

$$\zeta^{col} = F(\zeta^{col,N}) \tag{3.4}$$

for all $N \in \mathbb{N}$. The overall convention is that colours are used to track the similarities and discrepancies of ζ^{col} and $\zeta^{col,N}$: white particles live in both systems, red particles only in the N Colony System, and black particles only in the Collision Free System. The evolutions of these coloured populations depend on N and is thus recorded in $\zeta^{col,N}$.

The details are as follows:

1. *Let system ζ^{col} be a realisation of the Collision Free System on some probability space*

$$(\Omega, \mathcal{A}, \mathbb{P}) \tag{3.5}$$

as described in Definition 1.3. Assume that this probability space is rich enough to accommodate the following construction of the N Colony Systems.

2. *Assume that all particles of ζ^{col} are coloured white at any instant of time. Let for each N*

$$\zeta^{col,N} \tag{3.6}$$

be an identical copy of ζ^{col} that is enriched below with colours and additional particles. Let

$$K^{col,N}(t) \tag{3.7}$$

denote the number of colonies in system $\zeta^{col,N}$ carrying either white or red particles at time t. Similarly, let

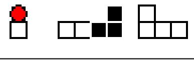
$$K^{col}(t) \tag{3.8}$$

denote the number of colonies in ζ^{col}. Let always any offspring obtain the colour of its parent particle.

3. *Now fix some N. If a migration event happens at time t that is caused by a white particle in system $\zeta^{col,N}$, a Bernoulli random variable with success probability*

$$\lim_{s \nearrow t} \frac{K^{col,N}(s)}{N} = \frac{K^{col,N}(t-)}{N} \tag{3.9}$$

is evaluated.



(a) *If the outcome is a success, the following changes are made to the evolution of* $\zeta^{col,N}$*: The newly inhabited colony is coloured black; this colour is adopted for the whole offspring of the migrating particle. Additionally, a red particle is added in one of the colonies that is occupied by either white or red particles. The colony is chosen uniformly at random from these colonies.*

(b) *If not, no change is made except for that all red particles that are located at the migration destination of the white particle is moved to an arbitrarily chosen free colony in* $\{1, ..., N\}$.

4. *Red particles evolve on top of system* ζ^{col} *(i. e. on the union of black and white particles) independently of this system in so far as the birth and migration mechanisms are concerned: Any single red particle gives birth at rate* s *and migrates at rate* c. *Red particles migrate freely on the first* N *colonies. (This may lead to colonies that carry a population of mixed black and red colours; but by convention, these particles do not interact with each other.)*

5. *The death rates of red particles depend on the number of white and red particles present at their specific location; namely, any pair of red particles located at the same colony coalesces at rate* d *to a red particle, and any pair of white and red particles coalesces at rate* d *to a white particle. A pair of red and black particles that is located at the same colony does not interact with each other.*

6. *Finally, introduce a* forced decoupling *if a migrating white particle would not collide according to the above rules, but the destined colony of the particle (which is by definition the colony to the right of the rightmost black or white colony) lies outside the allowed range*
$$\{1, ..., N\}\,. \tag{3.10}$$
If this happens, add a black and red particle pair into the system just as if the particle collided, with the modification that the red particle is placed onto a free colony within $\{1, ..., N\}$. *Free means occupied by neither red nor white particles; there must exist such a free colony because otherwise the particle would collide with probability* 1. *(A forced decoupling thus happens when single particle migrations cause gaps in the sequence of occupied colonies in system* ζ^{col}*).*

With this definition, the task to construct both the Collision Free System and the N Colony System on one probability space such that one can reuse the same realisation of ζ as basis for all ζ^N is accomplished. The simulations of the particle process that are presented throughout this diploma thesis rely on this stage of coupling. We summarize the relationships between the Multicolour Particle System and the original systems (recall Definition 1.12 on the term label-equivalent).

Proposition 3.2. *Let the mappings*

$$\pi_{W,B} \;:\; \mathbb{S}^{col} \;\to\; \mathbb{S}\,, \tag{3.11}$$
$$\pi^N_{W,R} \;:\; \mathbb{S}^{col} \;\to\; \mathbb{S}^N \tag{3.12}$$

reduce a given configuration of coloured particles to the corresponding configuration of uncoloured particles, when, in the case of $\pi_{W,B}$*, the red particles are ignored, and in the case of* $\pi^N_{W,R}$*, the black particles are ignored. (By definition, anything that lives beyond colony* N *is black). Consider the* $\mathbb{S}$*- and* $\mathbb{S}^N$*-valued processes*

$$(\zeta(t))_{t\geq 0} \text{ and } \{(\zeta^N(t))_{t\geq 0} \;:\; N \geq 1\} \tag{3.13}$$

as defined in Definition 1.3 and Definition 1.1. Consider also the $\mathbb{S}^{col}$*-valued Multicolour Particle Processes*

$$\{(\zeta^{col,N}(t))_{t\geq 0} \;:\; N \geq 1\}\,, \tag{3.14}$$

as defined in Definition 3.1.

1. *For any N, the Collision Free System ζ evolves like the white and black particles in $\zeta^{col,N}$, i. e.*
$$\zeta \stackrel{d}{=} \pi_{W,B}\zeta^{col,N} \text{ on } D([0,\ \infty),\ \mathbb{S}) . \tag{3.15}$$
Moreover, these realizations evolve identically for different values of N, i. e. for any $N \in \mathbb{N}$
$$\pi_{W,B}\zeta^{col,1} \equiv \pi_{W,B}\zeta^{col,N} \tag{3.16}$$
almost surely.

2. *For any N, the N Colony System ζ^N evolves, except for a random relabelling of colonies, like the white and red particles in $\zeta^{col,N}$, i. e. $\pi^N_{W,R}\zeta^{col,N}$ is a label-equivalent modification of ζ^N. In other words, there exists a time dependent random permutation $\eta(t)$ on $\{1,\ ...,\ N\}$ such that*
$$\zeta^N \stackrel{d}{=} (\pi^N_{W,R}\zeta^{col,N}) \circ \eta \text{ on } D([0,\ \infty),\ \mathbb{S}^N) . \tag{3.17}$$

Proof. These claims hold by definition. □

However, in order to tie the trajectories even more closely together amongst different values of N as well as amongst ζ^{col} and $\zeta^{col,N}$ for fixed N, we introduce additional rules. These will again make it necessary to enlarge the state space. Recall that in rule 3 of Definition 3.1 a coin is tossed that decides whether a migrating particle collides. The idea now is to use the same sequence of coins for all N. Since the success probabilities change with different values of N, the event of a success of a coin tossing is identified with the event
$$\{U \leq p\} , \tag{3.18}$$
when U is a $U[0,\ 1]$ random variable (that is, uniformly distributed on $[0,\ 1]$) and p is the success probability of the coin tossing. This construction allows to reuse the coin tossings even when the success probabilities change.

Definition 3.3 (Additional rules for the Multicolour Particle System)**.**
Assume that, additionally to the rules 1-6 of Definition 3.1, the following holds:

1. *Let $\{U_i\}_{i\in\mathbb{N}}$ be independent $U[0,\ 1]$-distributed random variables. In the N Colony System, colour the $(i+1)^{th}$ emerging site (due to the i^{th} migration of a white or black particle) black if*
$$U_i \leq \frac{K^{col,N}(t-)}{N} , \tag{3.19}$$
where t is the time of the i^{th} migration (cf. expression (3.9)). We refer to this colouring decision as N colouring.

2. *Identify black and red twin particles and couple their evolutions. This works as follows:*
   - *Give labels to the particles; give to black and red twins the same labels.*
   - *Couple birth and migration events of twin particles and give to their offspring the same labels.*
   - *Furthermore, to any pair of black particles that is located at the same colony at the time of birth of the younger particle, couple their (possible) coalescence event to the coalescence of the corresponding pair of red particles as long as the pair has not been separated by migration.*

   (For the red particles, we have to add additional exponential coalescence clocks for the red-white pairs as well as the red-red pairs that are no longer coupled to a black-black pair, e. g. if a coupled red-red-pair divides and meets again after some migration steps.)

Remark 3.4. *It will be shown in Lemma 3.5 that $K^N(t) \leq K(t)$ almost surely holds under the second additional rule. The first additional rule allows to give an almost sure bound on the difference $K(t) - K^N(t)$. This is done in Theorem 4.1. Occasionally, we will refine the system and colour sites black if*

$$U_i \leq \frac{i}{N}\,. \tag{3.20}$$

Since $K(t-) \leq i$ for t as above (equality holds if no single particle migrations happened before), this gives an upper bound on the number of black sites, provided we can ensure $K^N(t) \leq K(t)$ almost surely.

In summary, this construction embodies the N Colony System (counting both white and red particles) as well as the Collision Free System (counting both white and black particles) such that for different values of N the realisation of the Collision Free System does not change.

According to the construction described above, a collection of coupled N Colony Systems for different values of N is realized in two steps: Firstly, the CMJ process ζ and the sequence $\{U_i\}_{i\in\mathbb{N}}$ of independent uniformly distributed random variables determining the colourings are realized; and secondly, for each given N, the colourings are executed and red particles are added on top of the CMJ system.

3.3 Implications

It is intuitively clear that, due to collisions, in the N Colony System there should be less inhabited colonies than in the Collision Free System. The following shows that this is true in the coupling as introduced above.

Lemma 3.5. *Let $K^{col,N}(t)$ and $K^{col}(t)$ denote the number of occupied colonies in the N Colony System and in the Collision Free System respectively, when coupled as above. Let $T^{col,N}_{f(N)}$, $T^{col}_{f(N)}$ be their hitting times at level $f(N)$ for some non-negative function f. Then,*

$$K^{col,N}(t) \leq K^{col}(t) \tag{3.21}$$

and

$$T^{col}_{f(N)} \leq T^{col,N}_{f(N)} \tag{3.22}$$

hold almost surely.

Furthermore, if

$$\Pi^{col,N}(t),\ \Pi^{col}(t) \tag{3.23}$$

denote the total number of particles in the respective systems at time t, we have

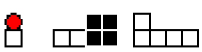
$$\Pi^{col,N}(t) \leq \Pi^{col}(t) \tag{3.24}$$

almost surely. Finally, the numbers of birth and migration events in the N Colony System are each almost surely dominated by the respective numbers in the Collision Free System.

Proof. The death of a black particle b leads always to the death of its red twin particle r: b coalesces with another black particle b' that did not migrate later than the birth time of b; hence, the death coupling of the pair $(b,\, b')$ to the pair $(r,\, r')$ is still intact, where r' is the twin of b', and the coalescence of $(b,\, b')$ triggers the coalescence of $(r,\, r')$.

Now, there are two events that may lead to an increase of $K^{col,N}(t)$ but not of $K^{col}(t)$:

- There might be a red particle r that migrates to an unoccupied colony while its black twin b does not migrate. This is excluded by the aforesaid and the coupling of migration events.



- Secondly, there might be a single white particle w with a stack of red particles on top. If now w migrates to a free colony, $K^{col,N}(t)$ increases by one while $K^{col}(t)$ remains constant. But the event only compensates a loss that $K^{col,N}(t)$ suffered earlier: At least one of the red particles on top of w has already migrated at least once; and each migration step of a red particle to an occupied colony as well as the conversion of a white particle to a red particle leaves $K^{col,N}(t)$ unchanged but increases $K^{col}(t)$. Again, this is ensured by the aforesaid and the coupling of migration events.

□

An immediate consequence of this lemma is the following convergence result.

Theorem 3.6 (convergence in the initial time window).
Consider the Multicolour Particle System ζ^{col}, $\zeta^{col,N}$.

1. *Let $\pi^N_{W,R}$ be defined as in Proposition 3.2, and*

$$\iota^N \ : \ \mathbb{S}^N \to \mathbb{S} \tag{3.25}$$

be the natural embedding of an N colony configuration into the space of infinite colony configurations that is obtained by filling up the remaining occupancy numbers with zeroes. Let

$$F(t) \equiv F(t,\ \{\pi_{W,B}\zeta^{col,1}_s \ : \ s \le t\}) \tag{3.26}$$

be a functional of the Collision Free System in its multicolour realisation and let

$$F^N(t) \equiv F(t,\ \{\iota^N \pi^N_{W,R}\zeta^{col,N}_s \ : \ s \le t\}) \tag{3.27}$$

be the corresponding functional of the N Colony System, both taking values in some Polish space E. Then,

$$\lim_{N\to\infty} (F^N(t))_{t\ge 0} = (F(t))_{t\ge 0} \tag{3.28}$$

almost surely uniformly on compact time intervals; and this implies almost sure convergence with respect to the Skorohod Topology on $D_E[0,\ \infty)$.

2. *In particular, convergence holds for the numbers of colonies*

$$(K^{col}(t),\ K^{col,N}(t))_{t\ge 0}\,, \tag{3.29}$$

the total numbers of particles in the system

$$(\Pi^{col}(t),\ \Pi^{col,N}(t))_{t\ge 0}\,, \tag{3.30}$$

and the statistics

$$(\Psi^{col}(t),\ \Psi^{col,N}(t))_{t\ge 0}\,, \tag{3.31}$$

the latter taking values in the Polish space $\mathcal{M}_{fin}(\mathbb{N})$.

Proof. We show for any $T > 0$ that the distance with respect to the metric on E vanishes, i. e.

$$\lim_{N\to\infty} \sup_{0\le t\le T} d_E(F^N(t) - F(t)) = 0\,. \tag{3.32}$$

This is trivially satisfied due to the additional coupling introduced in the first rule of Definition 3.3: For fixed ω and given T there is some N such that

$$\min(U_1,\ \ldots,\ U_{m(T)}) > \frac{m(T)}{N}\,, \tag{3.33}$$

where $m(T)$ denotes the number of migrations prior to time T. Using Lemma 3.5, the inequality (3.33) implies that for large N no colouring occurs up to time T and the two systems evolve identically on $[0,\ T]$. By Proposition 3.5.3 of [EK1986], (3.32) implies convergence with respect to the Skorohod Topology. □

3.4 Formalization

We sketch the formalization of the verbal description above, although we will refer throughout to the naive picture.

3.4.1 The state space

We need a definition of the state space of this coupled process, where we take into consideration the potential identification of red and black pairs. (Recall that this pairing is used to couple events between particles that live in the aftermath of a collision). If there were no identification of pairs, the following state space would be sufficient:

$$\mathbb{S}^{col} = \left\{ f \mid f \ : \ \mathbb{N} \to \mathbb{N}_0^3 \ , \ f(k) \neq (0,\ 0,\ 0) \text{ for finitely many } k \right\} . \tag{3.34}$$

The set $\mathbb{N}$ is used to index the colonies; the triple $f(k)$ then counts the white, red and black particles on colony k.

Now let the set $\mathbb{N}$ index the *particles* rather than the colonies, and collect for each particle the colony where it lives, its colour, and its coupled twin, if there is any. In order to simplify the description of the dynamics below, we index particles with $\mathbb{Z}$ instead of $\mathbb{N}$; it will be convenient to assign negative indices to red particles. This leads to the following definition.

Definition 3.7 (State space of the Multicolour Particle System)**.**

1. *Define the* state space for the N Colony System *via*

$$\mathbb{S}_{id}^{col,N} = \left\{ f \ \middle| \ \begin{array}{c} f \ : \ \mathbb{Z} \to \mathbb{N}_0 \times \{W,\ R,\ B\} \times \mathbb{N}_0 \ , \\ f(k) \neq (0,\ W,\ 0) \textit{ for finitely many } k \end{array} \right\} . \tag{3.35}$$

   Here, the first set $\mathbb{N}_0$ stands for the colony where the particle lives; this coordinate is zero if the particle is not alive. The set of symbols

$$\{W,\ R,\ B\} \tag{3.36}$$

   stands for white, red, and black, respectively. The third set $\mathbb{N}_0$ stands for the index of the twin particle, if there is any; if not, the third coordinate is zero. Hence, we say that, in state $f \in \mathbb{S}_{id}^{col,N}$, particle k is alive *if $f(k)(1) \neq 0$; we say it is* located in colony *m if $f(k)(1) = m$; we say that it has* colour *C, $C \in \{W,\ R,\ B\}$, if $f(k)(2) = C$; and we say that it is* twinned *with particle k' if $f(k)(3) = k'$.*

2. *Define the state space for the whole process via*

$$\mathbb{S}_{id}^{col} = [0,\ 1]^{\mathbb{N}} \times \prod_{N \in \mathbb{N}} \mathbb{S}_{id}^{col,N} . \tag{3.37}$$

   Here, the first component harbours the sequence $\{U_i \ : \ i \in \mathbb{N}\}$ that decides whether a migrating white particle collides.

Lemma 3.8. *The space $\mathbb{S}_{id}^{col}$ is Polish.*

Proof. This follows since $\mathbb{S}_{id}^{col}$ is the product of countably many Polish spaces. □

Finally, we note that the numbers of inhabitants of the i^{th} colony, ζ_i^{col} and $\zeta_i^{col,N}$, can be regained from the state $f \in \mathbb{S}_{id}^{col}$, as can be seen via the representations

$$\begin{aligned} \zeta_i^{col} &= \sum_{k \in \mathbb{N}} 1_{\{\pi^N f(k)(1) = i\}} 1_{\{\{\pi^N f(k)(2) \in \{W,\, B\}\}\}} , \\ \zeta_i^{col,N} &= \sum_{k \in \mathbb{N}} 1_{\{\pi^N f(k)(1) = i\}} 1_{\{\{\pi^N f(k)(2) \in \{W,\, R\}\}\}} , \end{aligned} \tag{3.38}$$

here, the projection $\pi^N \ : \ \mathbb{S}_{id}^{col} \to \mathbb{S}_{id}^{col,N}$ is used to obtain the state in the N Colony System. Similarly, we obtain the statistics Ψ and Ψ^N.

3.4.2 The dynamics

We see no advantage in repeating the complete set of rules as given in Definition 3.1 and Definition 3.3 and sketch thus only the general procedure. We take here the annealed point of view such that ζ^{col} and $\zeta^{col,N}$ evolve simultaneously in time.

- Initialize the first component of the system with a sequence $\{U_i\}$ of independent and uniformly on [0, 1] distributed random variables. Initialize the second component with empty configurations, i. e. choose the initial state $f \in \mathbb{S}^{col}_{id}$ given by

$$\begin{aligned} \pi^U f &= (U_i \,:\, i \in \mathbb{N})\,, \\ \pi^N f(1) &= (1,\, W,\, 0)\,, \\ \pi^N f(k) &= (0,\, W,\, 0)\,, \quad k \in \mathbb{Z} \setminus \{1\}\,, \end{aligned} \tag{3.39}$$

  for all $N \in \mathbb{N}$. The projection π^U gives the first component.

Proceed as follows:

- Assign to each $k \in \mathbb{Z}$ Poisson processes P^s_k, P^c_k of rate c and s respectively and to each pair k, $k' \in \mathbb{Z}$ Poisson processes $P^d_{k,k'}$ of rate d.
- In each N component, handle the events given by these processes according to Definitions 3.1 and 3.3. In particular, use for the k^{th} migration due to a white or black particle the variable U_k in order to decide whether the particle collides (this is has no effect on an already black particle but the rule ensures that the evaluation of the $\{U_k\}$ is done consistently amongst different values of N). The value of k can be obtained as the highest index of any colony that carries either black or white particles.
- When a red and black pair is produced, write the indices of the twin particles in the respective twin coordinates. Ignore the events of the red twinned particle except for additional death events and use instead the events of the coupled black particle.
- Assign negative indices to red particles. This ensures that the events for the white and black population are the same for all values of N.

We conclude with the remark that, by the non-explosion property (cf. Proposition 2.5), at any given instant of time only a finite number of these Poisson processes are evaluated; also, the coupling via the sequence $\{U_i\}$ ensures that, for any given time horizon $T > 0$, only a finite number of values of N have to be considered (cf. Theorem 3.6). This implies that the construction involving the Poisson processes is a well defined procedure.

Part II

Between the time windows

The goal of Part II is to bridge between the first and the second time window. Recall that on any bounded time interval $[0, t_0]$, $t_0 > 0$, the N Colony System eventually (for $N \to \infty$) follows the trajectory of the Collision Free System. On the other hand, collisions start to occur with probability $O(1)$ once time $T^N_{\epsilon N}$ is reached for any small $\epsilon > 0$; this is about where the beginning of the second time window is placed. Due to the increasing number of collisions, the trajectories then begin to separate. The evolution from time $T^N_{\epsilon N}$ onwards will be studied in detail in Part III.

The task is thus to understand the trajectories on the time interval $[0, T^N_{\epsilon N}]$. It is shown in Chapter 4 that, up to time $T^N_{\epsilon N}$, the systems ζ and ζ^N still evolve closely with small but noticeable distance. An almost sure bound is given for this distance that holds for large N.

Chapter 5 then shows that the large scale evolutions become deterministic once the initial time window is left. The statement is that the variances of the hitting times $T_{\epsilon N} - T_{\log N}$, $T^N_{\epsilon N} - T^N_{\log N}$ vanish for $N \to \infty$, at least in some weaker conditional sense.

4 An approximation bound

Simulations show that, in the multicolour coupling defined in Chapter 3 and for small $\epsilon > 0$, the N Colony System and the Collision Free System stay close to each other until about ϵN colonies are populated. This is shown in Figure 4.1. The goal of this chapter is to make this intuition precise.

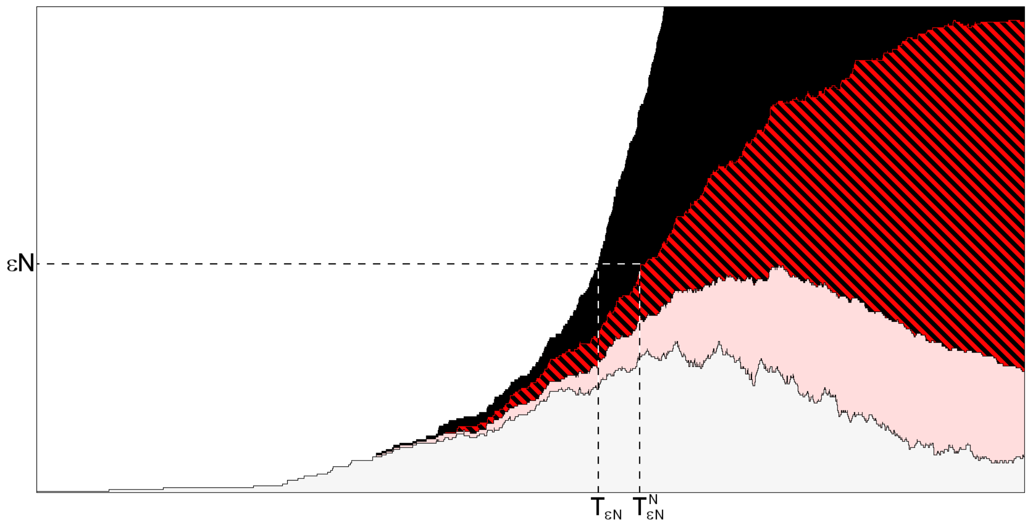


Figure 4.1: Plot of the trajectories of $K^{col,N}(t)$ (given by the transition between the red and black area) and $K^{col}(t)$ (given by the transition between the black area and the surroundings); time goes to the right. The light grey area indicates the amount of colonies that harbour only white particles, and the light red area corresponds to colonies that carry both white and red particles. The area that is striped in red and black corresponds to the number of colonies that contain red but no white particles, and finally the black area together with the striped area belongs to the colonies carrying black particles.

4.1 Statement of results

Motivated by simulations as seen in Figure 4.1, we wish to prove the following theorem, which states that the trajectories of the N Colony System and the Collision Free System are close up to time $T_{\epsilon N}$. This result is extended below in Corollary 4.14 to the time $T^N_{\epsilon N}$ which is larger than $T_{\epsilon N}$ (this ordering is guaranteed by Lemma 3.5).

Theorem 4.1 (Distance of trajectories)**.**
In the Multicolour Particle System $\zeta^{col,N}$, let $Z^N(t)$ denote the number of black sites at time t. Then, if ϵ is chosen small enough, the relative proportion of black colonies amongst black and white colonies at time $T^{col}_{\epsilon N}$ satisfies

$$\limsup_{N\to\infty} \frac{Z^N(T^{col}_{\epsilon N})}{\epsilon N} \le C\epsilon\,, \tag{4.1}$$

where the constant C does not depend on ϵ. The constant may be specified as

$$C = \sup_{t\ge 0} \mathbb{E}\left[W + w(t)\right] \cdot \left\lceil \frac{1+p_0}{p_0} \right\rceil^2 , \tag{4.2}$$

where p_0 denotes like in Proposition 2.14 the probability that the very first migrant is not single. The variables W, w determine together with the constant α the growth of the Collision Free System as described in Theorem 2.1.

Since the black particles give an upper bound on the distance of $\zeta^{col,N}$ and ζ^{col}, the following corollary is immediate.

Corollary 4.2. *For ϵ and C as above,*

$$\limsup_{N\to\infty} \frac{K^{col}(T^{col}_{\epsilon N}) - K^{col,N}(T^{col}_{\epsilon N})}{\epsilon N} \le C\epsilon\,. \tag{4.3}$$

Proof. Since $\zeta^{col,N}$ is given by the union of white and red particles and ζ^{col} is given by white and black particles, (4.3) follows from Theorem 4.1 and Lemma 3.5 which states that the black offspring is always larger than the red. □

We make the following convention in this chapter: It is tacitly understood that the systems $\zeta^{col,N}$, ζ^{col} are coupled via the multicolour coupling as introduced in Chapter 3. The superscript *col* that indicate the coupling of the systems is omitted.

Before getting involved with the proof, we quickly state the restriction on ϵ. This value must be small enough such that up to time $T_{\epsilon N}$ the migrants of the Collision Free System have not yet reached (with the help of single particle migrations) the colony with index N. Otherwise, the forced decoupling as described in the sixth rule of Definition 3.1 would come into effect; this would imply that even non-colliding migrants get coloured, and we would no longer be able to control the impacts of colourings.

Lemma 4.3. *There exists some $\epsilon_0 > 0$ and some finite random number $N_0(\epsilon_0)$ such that in the multicolour coupling no forced decoupling happens up to time $T_{\epsilon_0 N}$ for all $N \ge N_0(\epsilon_0)$. In particular, this holds for*

$$\epsilon_0 < \frac{p_0}{1+p_0}\ , \tag{4.4}$$

when p_0 is as in Proposition 2.14 the probability that the very first migrant is not single.

Proof. This follows from the sixth assertion of Proposition 2.14 which states that the number of migrations up to time $T_{\epsilon_0 N}$ does not exceed

$$\frac{1+p_0}{p_0}\epsilon_0 N\,. \tag{4.5}$$

□

4.2 The idea of proof

We consider the fraction on the left hand side of (4.1) and write down the following representation:

$$\frac{Z^N(T_{\epsilon N})}{K(T_{\epsilon N})} \approx \sum_{\substack{k \text{ such that} \\ \text{on the } k^{th} \text{ migration} \\ \text{a collision occurs}}} \frac{1}{k}\,. \tag{4.6}$$

When there are k colonies populated and one of it gets coloured, then the coloured proportion is k^{-1}; this proportion does not change in time because the $k-1$ competing forks grow at the same speed as the newly coloured one. (One may compare this with the blowing-up of a balloon on a rainy day: while the balloon gets blown up and the surface increases, raindrops of a fixed initial size hit the surface. The hitting rate is proportional to the surface. Between such hittings, the present wet area increases, but its proportion of the total surface does not change.)

This picture is inaccurate because it does not take into consideration the random fluctuations in the fork growths; the growth of competing forks is not even equal in distribution because of different internal initial conditions. Nevertheless, we prove the assertion for the caricature (4.6) first, because this will show the main arguments in a simple context. When doing so, we have to remind ourselves that k in (4.6) does not only range from 1 to $\lceil \epsilon N \rceil - 1$ but to a random higher number, because there will be more migrations than increases of $K(t)$ due to single particle migrations.

In a second step, we will take the random fork growths into consideration; this leads to

$$\frac{Z^N(T_{\epsilon N})}{K(T_{\epsilon N})} \approx \sum_{\substack{k \text{ such that} \\ \text{on the } k^{th} \text{ migration} \\ \text{a collision occurs}}} \frac{K^{(k)}(T_{\epsilon N} - \tau(k))}{\epsilon N}\,. \tag{4.7}$$

Here, the sequence

$$\left\{ (K^{(k)}(t))_{t \geq 0} \;:\; k \in \mathbb{N} \right\} \tag{4.8}$$

denotes i. i. d. copies of $(K(t))_{t\geq 0}$ (recall that any fork grows just like the whole tree), and $\tau(k)$ is the time of the k^{th} migration. (This is still not an equality, because we have to avoid double-counting of coloured forks; that is if a descendant of a black particle gets coloured again, we must not count its offspring again. On the other hand, this possible self-containment destroys the i. i. d. property.)

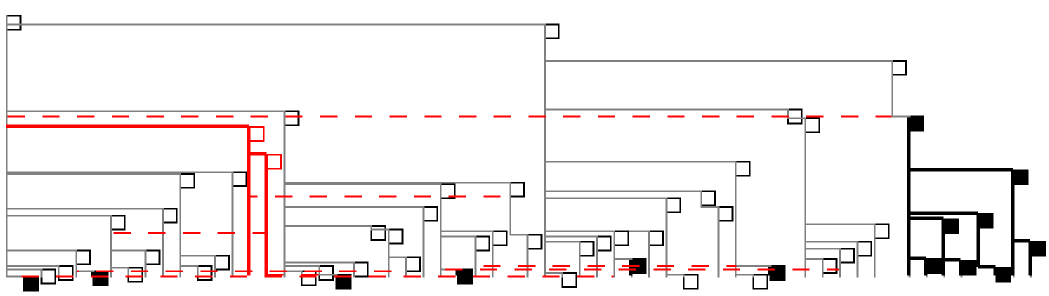


Figure 4.2: The migration tree as seen in Figure 2.1 modified in order to incorporate the N Colony System. The value of N is 100. Circular migration of colliding particles is plotted with dashed lines. In view of Theorem 4.1, the task is to control both the number of appearances of black forks (which can be handled independently of the concrete shape of the tree) and the growth of these forks.

Remark 4.4. *We may convince ourselves on an informal level that* (4.7) *could work: Assuming that close to time* 0*, say up to time* $T_{\log N}$*, there is no colouring, we expect the tree growth to be rather smooth in the time window of interest; and we write down the following integral approximation:*

$$\frac{Z^N(T_{\epsilon N})}{\epsilon N} \quad \approx \quad \frac{1}{\epsilon N}\int_{T_{\log N}}^{T_{\epsilon N}} K^{(s)}(T_{\epsilon N}-s)\frac{K(s)}{N}SK(s)\alpha ds\,. \tag{4.9}$$

The integral can be explained from right to left: The total number of new colonies in time interval $[s,\ s+ds)$ *is about* $K(s)\alpha ds$*; and for each new colony there are in average about* S *migrations necessary, where* $S \geq 1$ *is some constant (that clearly is somehow related to the sequence* $(s(i))_{i\geq 1}$ *that counts the single particle migrations). A fraction of*

$$\frac{K(s)}{N} \tag{4.10}$$

of these migrations gets coloured black (actually, the fraction would be $N^{-1}K^N(s)$*, but we may use* $K^N(s) \leq K(s)$ *by virtue of Lemma 3.5). The expression*

$$N^{-1}S(K(s))^2\alpha ds \tag{4.11}$$

thus gives the production rate of newly coloured migrations; and the term

$$K^{(s)}(T_{\epsilon N}-s) \tag{4.12}$$

indicates the size of the offspring that stems from such a newly coloured particle.

Using the growth properties of $K(t)$ *that are described in Theorem 2.1, and assuming a law of large numbers on the integrand* $K^{(s)}(T_{\epsilon N}-s)$ *on small time intervals, we obtain*

$$K^{(s)}(T_{\epsilon N}-s)ds \approx \mathbb{E}\left[W\right]e^{\alpha(T_{\epsilon N}-s)}ds \approx \mathbb{E}\left[W\right]e^{-\alpha s}\frac{\epsilon N}{W}ds\,, \tag{4.13}$$

where the factor $W^{-1}\epsilon N$ *stems from the CMJ Theorem applied to* $\exp(\alpha T_{\epsilon N})$*. Similarly,* $K(s) \approx We^{\alpha s}$*; hence*

$$\frac{Z^N(T_{\epsilon N})}{\epsilon N} \approx \frac{1}{\epsilon N}\mathbb{E}\left[W\right]\frac{\epsilon N}{W}\left[SW^2e^{\alpha s+\alpha s-\alpha s}\right]_{s=T_{\log N}}^{s=T_{\epsilon N}} = \frac{S\mathbb{E}\left[W\right]}{\epsilon N}\frac{\epsilon N}{N}\left(\epsilon N-\log N\right)\,, \tag{4.14}$$

and this is ϵ *multiplied with some constant.*

We first prove the claim rigorously for the caricature (4.6) in the next chapter before the actual proof of the Theorem is presented, starting from presentation (4.7). In order to ease the notation slightly, we let the sums in the following range to $\lceil \epsilon N \rceil$ instead of $\lceil \epsilon N \rceil - 1$ which would be the correct value.

4.3 A caricature: A deterministically growing tree

We want to show that expression (4.6) is small. Recall that the k^{th} migrant gets coloured if

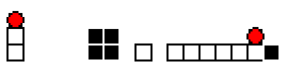
$$U_k \leq \frac{K}{N}\,, \tag{4.15}$$

when K is the number of populated colonies in the N Colony System prior to that migration event. Here, the U_k are i. i. d. random variables that are uniformly distributed on the unit interval. (We use in the following caricature calculation for convenience an array $(U_{i,k})_{i,k\in\mathbb{N}}$ instead.)

Recall also the bound

$$K^N(t) \leq K(t) \tag{4.16}$$

(cf. Lemma 3.5). Looking for an upper bound on the number of black sites, (4.16) implies that we may replace K in the right hand side of (4.15) by the number of colonies in the Collision Free System. This number in turn is bounded from above by k, the number of migrations.



Proposition 4.5. *Fix some $\epsilon > 0$. For $i \in \mathbb{N}$, let as before $s(i)$ denote the number of migrations that are necessary to increase $K(t)$ from i to $i+1$. Consider an array $(U_{i,k})_{i,k\in\mathbb{N}}$ of independent random variables that are uniformly distributed on the unit interval. Assume that these variables are independent of the $s(i)$. Define*

$$P^N = \sum_{i=1}^{\lceil \epsilon N \rceil} \sum_{k=1}^{s(i)} \frac{1}{i} 1_{\{U_{i,k} \leq i/N\}} . \tag{4.17}$$

Then,

$$\mathbb{E}\left[P^N\right] \leq \epsilon \mathbb{E}\left[s(1)\right] + o(1) \tag{4.18}$$

for $N \to \infty$ and furthermore

$$P^N \leq \epsilon \mathbb{E}\left[s(1)\right] + o(1) \tag{4.19}$$

almost surely.

Remark 4.6. *The expression P^N equals $P^N(M)$ evaluated at $M = \lceil \epsilon N \rceil$, where $P^N(M)$ is defined as*

$$P^N(M) = \sum_{i=1}^{M} \sum_{k=1}^{s(i)} \frac{1}{i} 1_{\{U_{i,k} \leq i/N\}} . \tag{4.20}$$

This is a function of N and M, and for fixed M the mapping

$$N \mapsto P^N(M) \tag{4.21}$$

is monotonically decreasing due to the coupling of colouring events via the array $(U_{i,k})_{i,k\in\mathbb{N}}$. Meanwhile, for fixed N, the mapping

$$M \mapsto P^N(M) \tag{4.22}$$

is monotonically increasing, because nonnegative summands are added atop while the previous summands remain unchanged. The Proposition states that along the diagonal

$$M(N) = \lceil \epsilon N \rceil \tag{4.23}$$

both effects are in balance.

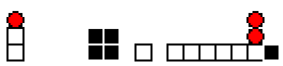
Proof. First, use Proposition 2.14 to bound the expectation of P^N:

$$\mathbb{E}\left[P^N\right] = \sum_{i=1}^{\lceil \epsilon N \rceil} \mathbb{E}\left[s(i)\right] \frac{1}{i} \frac{i}{N} \leq \frac{\lceil \epsilon N \rceil}{N} \mathbb{E}\left[s(1)\right] . \tag{4.24}$$

Fix some $\eta > 0$ and consider the i. i. d. sequence $\{s^{(i)}(1) \,:\, i \in \mathbb{N}\}$ that has been introduced in Proposition 2.14. Recall that the second moment of a Bernoulli random variable is given by its parameter. Recall also the Blackwell-Girshick formula for the variance of a random sum.

By Tchebychev's Inequality,

$$\begin{aligned} \mathbb{P}\left(P^N - \epsilon \mathbb{E}\left[s(1)\right] > \eta\right) &\leq \mathbb{P}\left(\left|\sum_{i=1}^{\lceil \epsilon N \rceil} \sum_{k=1}^{s^{(i)}(1)} \frac{1}{i} 1_{\{U_{i,k} \leq i/N\}} - \mathbb{E}\left[P^N\right]\right| > \eta\right) \\ &\leq \frac{1}{\eta^2} \sum_{i=1}^{\lceil \epsilon N \rceil} \left(\mathbb{V}\mathrm{ar}\left[s^{(i)}(1)\right] + \mathbb{E}\left[s^{(i)}(1)\right]\right) \left(\frac{1}{i}\right)^2 \left(\frac{i}{N}\right) \\ &\leq 2 \frac{\mathbb{V}\mathrm{ar}\left[s(1)\right] + \mathbb{E}\left[s(1)\right]}{\eta^2} \frac{\log\left(\epsilon N\right)}{N} ; \end{aligned} \tag{4.25}$$



here, the usual approximation of the partial harmonic sum by the natural logarithm was used. This implies that the subsequence

$$\left(P^{N^2}\right)_{N\geq 1} \tag{4.26}$$

is asymptotically bounded by $\epsilon\mathbb{E}\left[s(1)\right]$. We have to make sure, using essentially the same calculation, that P^N does not get large between N^2 and $(N+1)^2$. There are two possibilities for P^N to change: First, there may be a perceivable number of forks that are coloured in the N^2-tree but that are cut off in the course of the application of the additional colour coupling, i. e. when

$$\frac{i}{(N+1)^2} < U_{i,k} \leq \frac{i}{N^2}\,. \tag{4.27}$$

Secondly, the new indices that are added atop may add to the proportion. Only the second possibility actually increases P^N, so only this has to be considered.

Abbreviate for $N,\ k \in \mathbb{N}$

$$A(N^2,\, i,\, k) \quad = \quad \left\{U_{i,k} \leq \frac{i}{N^2}\right\}\,, \tag{4.28}$$

and use that

$$\mathbb{E}\left[\sum_{i=\lceil\epsilon N^2\rceil+1}^{\lceil\epsilon(N+1)^2\rceil}\ \sum_{k=1}^{s^{(i)}(1)} \frac{1}{i} 1_{A(N^2,i,k)}\right] \leq \epsilon\frac{\left[(N+1)^2 - N^2\right]}{N^2}\mathbb{E}\left[s(1)\right] = O(N^{-1}) \tag{4.29}$$

for $N \to \infty$. Then, for arbitrary $\eta > 0$, using $s(i) \leq s^{(i)}(1)$ where the right hand side is independent amongst different values of i (recall Proposition 2.14),

$$\begin{aligned}\mathbb{P}\left(\sup_{M\in I(N^2)} P^M - P^{N^2} > \eta\right) &\leq \mathbb{P}\left(\sum_{i=\lceil\epsilon N^2\rceil+1}^{\lceil\epsilon(N+1)^2\rceil}\ \sum_{k=1}^{s(i)} \frac{1}{i} 1_{A(N^2,i,k)} > \eta\right)\\ &\leq \frac{\mathbb{V}\mathrm{ar}\left[s(1)\right] + \mathbb{E}\left[s(1)\right]}{\left(\eta - O(N^{-1})\right)^2} \sum_{i=\lceil\epsilon N^2\rceil+1}^{\lceil\epsilon(N+1)^2\rceil} \left(\frac{1}{i}\right)^2 \frac{i}{N^2}\\ &\leq \frac{\mathbb{V}\mathrm{ar}\left[s(1)\right] + \mathbb{E}\left[s(1)\right]}{\left(\eta - O(N^{-1})\right)^2}\left[\frac{\log(N+1) - \log N}{N^2}\right]\,, \end{aligned} \tag{4.30}$$

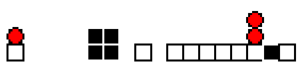
and this is summable. Hence, for any η and large N, the trajectory of $N \mapsto P^N$ stays in an η-envelope around the subsequence $N \mapsto P^{N^2}$ which is known to be bounded by $\epsilon\mathbb{E}\left[s(1)\right]$. This proves the proposition. □

In the actual proof of Theorem 4.1, the approach will be quite similar: First, convergence along a convenient subsequence is shown, and secondly, the coupling of different values of N is exploited to show that between the indices of the subsequence nothing unexpected happens. The tools will again be the Borel Cantelli Lemma and Tchebychev's inequality, which explains why we need to impose a restriction on the second moment of the tree growth.

Additionally, in order to break up several dependencies amongst the summands (counting fork sizes) and the $s(i)$ (counting single particle migrations), there are more technical steps to be taken below: Both the number of single particle migrations and the fork growth time

$$T_{\epsilon N} - \tau(k) \tag{4.31}$$

will be replaced by deterministic upper bounds. The first is justified by Proposition 2.14, while the latter requires that the tree growth is sufficiently smooth and thus works only when being far away from time 0.



Hence, we have to examine separately a first time window starting at time 0 where almost no (but almost surely some) black offspring is produced. This is done below in Proposition 4.7 (about the number of initial colourings) and Lemma 4.9 (about the offspring of these coloured particles). Lemma 4.12 then considers the time window where the above mentioned deterministic replacements are applicable. This will finish the proof.

4.4 Preparation of proof: Examination of colouring events

The following is an assertion on the evolution of colourings in time. Only the first claim will be used further below; the other assertions are meant to draw a more detailed picture of the appearances of black forks. The proof of these additional assertions can be found in Appendix B.1.

The earlier a colouring occurs, the more it contributes to the black proportion; if one wants to prove that the painted proportion is small, one must ensure that there are only few events in the beginning. In the following, we examine the asymptotic distribution of colouring events on the set of indices. The assertion is that the only coloured forks that are asymptotically visible are those to the very right.

Proposition 4.7. *Fix some* $A \in \mathbb{N}$, $\beta \in (0,\ \frac{1}{4})$, $\delta \in (0,\ \frac{1}{2})$, $\epsilon \in (0,\ A^{-1})$ *and* $p_0 \in (0,\ 1)$. *For* $N \in \mathbb{N}$ *consider the set of indices*

$$I^N = \{1,\, 2,\, ...,\, \lceil \epsilon N \rceil \cdot A\}\,. \tag{4.32}$$

Assume that each index $i \in I^N$ *is coloured independently of each other with probability*

$$\frac{i}{N}\,. \tag{4.33}$$

For $a,\ b \in I^N$ *let the quantity* $C^N(a,\, b)$ *denote the set of coloured indices* i *with* $a \le i < b$. *Then, there exists some*

$$N_0 \equiv N_0(\omega;\, A,\, \beta,\, \delta,\, \epsilon,\, p_0) \in \mathbb{N} \tag{4.34}$$

such that C^N *has the following properties for all* $N \ge N_0$*:*

1. *There are almost no colourings up to index* $\left\lceil N^{\frac{1}{4}-\beta} \right\rceil$*; more precisely*

   $$\#C^N\left(1,\, \left\lceil N^{\frac{1}{4}-\beta} \right\rceil\right) = 0 \text{ or } 1\,. \tag{4.35}$$

   This bound is sharp in the sense that, if the colourings are independent amongst different values of N,

   $$\#C^N\left(1,\, \left\lceil N^{\frac{1}{4}} \right\rceil\right) > 1 \tag{4.36}$$

   happens for infinitely many values of N.

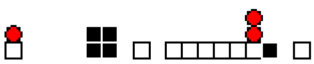
2. *There are no more than* $(\log N)^2$ *colourings between index* $\left\lceil N^{\frac{1}{4}-\beta} \right\rceil$ *and index* $\left\lceil N^{\frac{1}{2}} \right\rceil$*:*

   $$\#C^N\left(\left\lceil N^{\frac{1}{4}-\beta} \right\rceil,\, \left\lceil N^{\frac{1}{2}} \right\rceil\right) \le (\log N)^2\,. \tag{4.37}$$

3. *There are no more than* $2N^{1-2\delta}$ *colourings between* $\left\lceil N^{\frac{1}{2}} \right\rceil$ *and* $\lceil N^{1-\delta} \rceil$*:*

   $$\#C^N\left(\left\lceil N^{\frac{1}{2}} \right\rceil,\, \lceil N^{1-\delta} \rceil\right) \le 2N^{1-2\delta}\,. \tag{4.38}$$

   The last assertion remains true if $\delta \equiv \delta(N)$ *depends on* N *and is bounded away from* $\frac{1}{2}$.



4. *There are no more than* $2(A\epsilon)^2 N$ *colourings between* $\lceil N^{1-\delta} \rceil$ *and* $A \lceil \epsilon N \rceil$:

$$\# C^N \left(\lceil N^{1-\delta} \rceil , A \lceil \epsilon N \rceil \right) \leq 2(A\epsilon)^2 N \,. \tag{4.39}$$

Remark 4.8. *In order to make* (4.6) *small, i. e.*

$$\sum_{k \in C^N(1,\, A\lceil \epsilon N \rceil)} \frac{1}{k} \leq \epsilon \cdot const \,, \tag{4.40}$$

it would be sufficient to have a finite partition I_1^N, ..., I_m^N *of the set of indices* I^N *such that for all* k

$$\lim_{N \to \infty} \frac{\# C^N(I_k^N)}{\# I_k^N} = 0 \,, \tag{4.41}$$

where $C^N(I_k^N)$ *denotes the coloured subset of* I_k^N. *This corresponds to the idea that shifting all colouring events to the very left of the interval* I_k^N *gives a larger proportion. If then the coloured indices are still outweighed by the noncoloured, the contribution to* (4.7) *is asymptotically negligible.*

This was the motivation behind the present Proposition in the first place. The outcome is that one needs an additional argument for the first interval up to index $\lceil N^{\frac{1}{4}} \rceil$, *while the sketched argument works for the second (up to index* $\lceil N^{\frac{1}{2}} \rceil$*) and for the third, as long as*

$$1 - 2\delta < \frac{1}{2} \,; \tag{4.42}$$

e. g. for $\delta = \frac{1}{3}$. *Even more, one can repeat the argument a finite number of times, getting arbitrarily close to power* 1*: that is, the sum* (4.7) *is* $o(1)$ *when summed up to* $N^{1-\delta}$, *for any prefixed* $\delta \in (0,\ 1)$. *This supports the intuition that only the indices that are very close to* $\lceil \epsilon N \rceil$ *contribute to* (4.7)*, because only there the colouring probability is* $O(1)$.

Proof of the first assertion. In order to obtain an upper bound, colour particles with probability

$$p = \frac{\lceil N^{\frac{1}{4}-\beta} \rceil}{N} \,. \tag{4.43}$$

Define

$$f(N) = \left\lceil N^{\frac{1}{4}-\beta} \right\rceil \,. \tag{4.44}$$

Then,

$$\begin{aligned} \mathbb{P}\left(\# C^N(1,\, f(N)) > 1 \right) &= 1 - (1-p)^{f(N)} - f(N)p(1-p)^{f(N)-1} \\ &= 1 - (1-p)^{f(N)-1} \left[(1-p) + f(N)p \right] \,. \end{aligned} \tag{4.45}$$

Using

$$\begin{aligned} (1-p)^{f(N)-1} &= \exp\left[(f(N)-1) \log(1-p) \right] \\ &\geq \exp\left[-(f(N)-1)\left(p+p^2\right) \right] \\ &\geq 1 - (f(N)-1)\left(p+p^2\right) \,, \end{aligned} \tag{4.46}$$

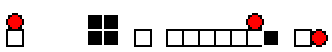
we obtain

$$\begin{aligned} \mathbb{P}\left(\# C^N(1,\, f(N)) > 1 \right) &\leq 1 - \left[1 - (f(N)-1)\left(p+p^2\right) \right] \left[1 + p\left(f(N)-1\right) \right] \\ &= p^2 \left(f(N)-1\right)^2 + O(p^2) \,, \end{aligned} \tag{4.47}$$

which is of order $N^{-1-4\beta}$ and thus summable.

Since the consideration of the true colouring probabilities iN^{-1} only improves the bounds by a factor of $\frac{1}{2}$, the inequalities may be reversed, using different constants; and the second claim follows when considering the colourings up to index $\lceil N^{\frac{1}{4}} \rceil$. □

The proof of the remaining assertions can be found in Appendix B.1.



4.5 Proof of Theorem 4.1

The proof is divided into two parts: Subsection 4.5.1 handles the black forks that appear somewhere in the time interval $[0,\ T_{N^{\frac{1}{5}}})$ where only few colourings happen. Here, $T_{N^{\frac{1}{5}}}$ is a rather arbitrary time that is smaller than $T_{N^{\frac{1}{4}}}$, for which Proposition 4.7 gave a bound on the number of colourings. Subsection 4.5.2 then considers the time interval $[T_{N^{\frac{1}{5}}},\ T_{\epsilon N})$ where more colourings happen but the tree growth is smoothed by law of large number effects. The analysis is complicated by the fact that there are more migrations than increases of $K(t)$ due to single particle migrations. We will thus make frequent use of Proposition 2.14.

4.5.1 Proof in the first time window $[0,\ T_{N^{\frac{1}{5}}})$

We first show that up to time $T_{N^{\frac{1}{5}}}$ there is no noticeable deviation in the trajectories: At most one black particle is observed whose offspring is negligible.

Lemma 4.9. *Let $Z^{N,1}(t)$ be the number of black sites at time t that have an ancestor that changed from white to black somewhere in the time interval*

$$[0,\ T_{N^{\frac{1}{5}}})\,. \tag{4.48}$$

Then,

$$\limsup_{N\to\infty} \frac{Z^{N,1}(T_{\epsilon N})}{K(T_{\epsilon N})} = 0\,. \tag{4.49}$$

In the spirit of some variants of the proof of the law of large numbers, the proof is split into two parts: First, convergence is shown along some "nice" subsequence $R \mapsto N(R)$; thereafter, convergence is shown to hold for the whole sequence.

Proof (part 1: Convergence along a subsequence). Fix some small $\beta \in (0, 1/20)$. Proposition 2.14 implies

$$T_{N^{\frac{1}{5}}} \le T_{\frac{p_0}{1+p_0}N^{\frac{1}{4}-\beta}} \le \tau(\lceil N^{\frac{1}{4}-\beta}\rceil)\,, \tag{4.50}$$

where again $\tau(M)$ denotes the time of the M^{th} migration. Hence, there happen not more than $\lceil N^{\frac{1}{4}-\beta}\rceil$ migrations in the considered time interval. Proposition 4.7 is thus applicable and yields that at most one colouring is observed for large N. Thus, the task is to show that the offspring of this single particle is negligible.

By coupling via the sequence $(U_k)_{k\ge 1}$, the number of colourings up to time t is decreasing for fixed t. Define for $R \in \mathbb{N}$

$$N(R) = \inf\{N\ :\ \text{migrations } \{1,\ \dots,\ R\} \text{ did not get coloured under } N \text{ colouring}\} \tag{4.51}$$

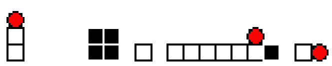
and let $\rho(N)$ be the index of the first migration that gets coloured under N colouring. By definition, $\rho(N) > R$ if $N \ge N(R)$. We show that along the subsequence $R \mapsto N(R)$ the black proportion vanishes.

For given large R and $N \equiv N(R)$, $\rho \equiv \rho(N(R))$, the only contribution to the black population at time $T_{\epsilon N}$ is made by the fork rooted at ρ. This fork evolves in time just like the whole tree, at least in distribution. Hence, there exists a copy $K^{(\rho)}(\cdot)$ of $K(\cdot)$ such that the contribution is given by

$$Z^{N,1}(T_{\epsilon N}) = K^{(\rho)}(T_{\epsilon N} - \tau(\rho))\,. \tag{4.52}$$

In order to get rid of the random times, use the monotonicity of $K^{(\rho)}(\cdot)$ to bound as follows:

$$\limsup_{R\to\infty} \left[K^{(\rho(R))}(T_{\epsilon N(R)} - \tau(\rho(R))) - K^{(\rho(R))}(f(N(R),\ R))\right] \le 0\,. \tag{4.53}$$



Here, $f(N,\, R)$ is a deterministic function that is an upper bond for $T_{\epsilon N} - \tau(\rho)$, at least for large R. There exists some $A < 1$ such that we can choose

$$f(N,\, R) = \frac{1}{\alpha}\left(\log\frac{\epsilon N}{AR} + \log\log R\right) , \tag{4.54}$$

for the following reasons: Clearly, $\tau(\rho) \geq \tau(R+1)$, and using Proposition 2.14, for large R and small A

$$\tau(R+1) \geq T_{A(R+1)} . \tag{4.55}$$

Finally, the difference of the hitting times

$$T_{\epsilon N(R)} - T_{A(R+1)} \tag{4.56}$$

can be bounded by (4.54) because the stochastic error term is eventually outdistanced by the $\log\log R$ term (recall Corollary 2.11 for the precise expression for (4.56)).

Now, we are ready to turn to (4.49), when $N \equiv N(R)$. If R is large enough, it follows that

$$\begin{aligned} \frac{Z^{N,1}(T_{\epsilon N})}{K(T_{\epsilon N})} &\leq \frac{K^{(\rho)}(f(N,\ R))}{\epsilon N(R)} \\ &\leq \left(W^{(\rho)} + w^{(\rho)}(f(N,\ R))\right)\frac{\log R}{AR} . \end{aligned} \tag{4.57}$$

Convergence of $(W^{(\rho)}(AR)^{-1}\log R)$ to 0 follows from existence of its second moment; and we will show convergence of $w^{(\rho)}(f(N(R),\, R))(AR)^{-1}\log R$ similarly. The variable $f(N(R),\, R)$ is (U_i)-measurable and thus in particular independent of the variables $w^{(\rho)}$; for fixed n, $f(n,\, R)$ is deterministic. Hence,

$$\begin{aligned} \mathbb{P}\left(\left|w^{(\rho)}(f(N(R),\, R))\right|\frac{\log R}{AR} > \eta\right) &\leq \left(\frac{\log R}{\eta AR}\right)^2 \mathbb{E}\left[\mathbb{E}\left[\left|w^{(\rho)}(f(N(R),\, R))\right|^2 \mid N(R)\right]\right] \\ &\leq \left(\frac{\log R}{\eta AR}\right)^2 \sup_{s\geq 0}\mathbb{E}\left[\left|w^{(\rho)}(s)\right|^2\right] ; \end{aligned} \tag{4.58}$$

this is finite when summed over R. □

Next, we need to consider more general subsequences.

Proof (part 2: Convergence along any subsequence). Now let $(N_r)_r$ be a $(U_i)_{i\geq 1}$-measurable subsequence with $N_r \to \infty$ and with $N_r = N(r) + k_r$ such that $N_r < N(s)$ for all $s > r$ with $N(s) \neq N(r)$. Conditionally on the sequence $(U_i)_{i\geq 1}$, the calculation may be executed as in equation (4.58), yielding a. s. convergence to 0. In other words, one may choose any N_r between $N(r)$ and $N(r+1)$ and obtains a sequence converging to 0; in particular, for fixed ω, also the (not $(U_i)_{i\geq 1}$-measurable) sequence $(N_r)_r$ that attains for each r the maximum in the r^{th} index interval, i. e.

$$N_r(\omega) = \underset{N\,:\,N(r)\leq N\leq N(r+1)}{\operatorname{argmax}} \frac{Z^{N,1}(T_{\epsilon N})(\omega)}{\epsilon N} , \tag{4.59}$$

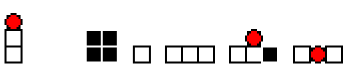
gives a zero sequence. This implies a. s. convergence of (4.49) to zero along any deterministic subsequence. □

Remark 4.10. *In the proof, a deterministic upper bound for the times*

$$T_{\epsilon N} - T_R \tag{4.60}$$

has been used, and this allowed to bound with

$$\sup_{s\geq 0}\mathbb{E}\left[\left|w^{(\rho)}(s)\right|^2\right] . \tag{4.61}$$



If one took the original random times, one had to deal with

$$\mathbb{E}\left[\left|\sup_{s\geq 0} w^{(\rho)}(s)\right|^2\right] , \tag{4.62}$$

which seems considerably harder.

4.5.2 Proof in the second time window $[T_{N^{\frac{1}{5}}},\ T_{\epsilon N})$

We introduce the quantity P^N that lies above $(\epsilon N)^{-1} Z^N(T_{\epsilon N})$. For this quantity, we prove convergence below in Lemma 4.12.

Lemma 4.11. *Let $Z^{N,2}(t)$ be the number of black sites at time t that have an ancestor that changed from white to black somewhere in the time interval*

$$[T_{N^{\frac{1}{5}}},\ T_{\epsilon N}) . \tag{4.63}$$

Define

$$P^N = \sum_{k \in I^N} 1_{\{U_k \leq k/N\}} \frac{K^{(k)}(f(N,\ k))}{\epsilon N} ,$$

where the $K^{(k)}(\cdot)$ are i. i. d. copies of $K(\cdot)$. The sum is indexed with the set

$$I^N = \left\{\left\lceil N^{\frac{1}{5}}\right\rceil, ..., A\left\lceil \epsilon N\right\rceil\right\} , \tag{4.64}$$

and

$$A = \left\lceil \frac{1+p_0}{p_0} \right\rceil . \tag{4.65}$$

Then, there exists a sequence of random variables $\{\tilde{P}^N\}$ such that the following almost sure bound holds for large N:

$$\frac{Z^{N,2}(T_{\epsilon N})}{K(T_{\epsilon N})} \leq \tilde{P}^N . \tag{4.66}$$

Moreover, for any $M > N$,

$$\begin{aligned} \tilde{P}^N &\leq_{st} P^N , \\ \tilde{P}^M - \tilde{P}^N &\leq_{st} P^M - P^N . \end{aligned} \tag{4.67}$$

The approximation is done in multiple steps.

Step 1: Replacement of the $s(i)$ by their upper bound. By Proposition 2.14, the number of migrations up to time $T_{\epsilon N}$ is bounded from above by

$$A\left\lceil \epsilon N\right\rceil \tag{4.68}$$

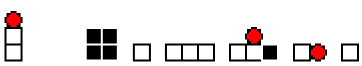
if N is chosen large enough. Thus, for all N greater than some $N_0(\omega)$,

$$\frac{Z^{N,2}(T_{\epsilon N})}{K(T_{\epsilon N})} \leq \sum_{k \in I^N \cap B^N} 1_{\{U_k \leq k/N\}} \frac{\tilde{K}^{(k)}(T_{\epsilon N} - \tau(k))}{\epsilon N} , \tag{4.69}$$

where the approximation only introduces additional summands. The process $\tilde{K}^{(k)}(\cdot)$ counts the colony offspring of the k^{th} migration, and the random set

$$B^N \subset I^N \tag{4.70}$$

sieves out the indices that lie in the offspring of already coloured colonies. □



Step 2: Removal of the random times. By monotonicity of $K(\cdot)$, imposing an upper bound on the random times $T_{\epsilon N} - \tau(k)$ only increases the left hand side of (4.67). Again by Proposition 2.14, we have for A as above and for k large enough,

$$T_{A^{-1}k} \leq \tau(k)\,. \tag{4.71}$$

We may thus define for fixed $\eta > 1$

$$f(N,\ k) = \frac{1}{\alpha}\log\frac{\epsilon N}{A^{-1}k} + \frac{\log\eta}{\alpha}\,; \tag{4.72}$$

this is an upper bound because for the random times it is known that

$$T_{\epsilon N} - T_{A^{-1}k} = \frac{1}{\alpha}\log\frac{\epsilon N}{A^{-1}k} + \frac{1}{\alpha}\log\frac{W + w(T_{A^{-1}k})}{W + w(T_{\epsilon N})}\,, \tag{4.73}$$

where the second summand converges to 0 for $N,\ k \to \infty$. Hence, for large N,

$$\frac{Z^{N,2}(T_{\epsilon N})}{K(T_{\epsilon N})} \leq \sum_{k\in I^N\cap B^N} 1_{\{U_k\leq k/N\}}\frac{\tilde{K}^{(k)}(f(N,\ k))}{\epsilon N} \tag{4.74}$$

almost surely. □

Step 3: Introducing i. i. d. forks. The next step is to replace in (4.74) the family $\tilde{K}^{(m)}(\cdot)$ that depends on the original tree by a family of i. i. d. copies. The obtained bound holds then only stochastically instead of almost surely. This also removes the restriction to the random set B^N. The obtained bound equals (4.67). □

We are now ready to finish the proof of Theorem 4.1. This is done by showing that the quantity P^N as introduced in Lemma 4.11 converges almost surely.

Lemma 4.12. *Let P^N be as in Lemma 4.11 and C as in Theorem 4.1. Then, there exists some positive constant $\tilde{C} \leq C\epsilon$ such that for any $\delta > 0$*

$$\sum_{N\geq 1}\mathbb{P}\left(\left|P^{N^2} - \tilde{C}\right| > \delta\right) + \sum_{N\geq 1}\mathbb{P}\left(\max_{N^2<M\leq(N+1)^2}(P^M - P^{N^2}) > \delta\right) < \infty\,. \tag{4.75}$$

In particular,

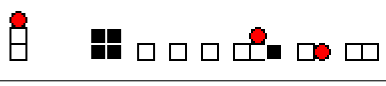
$$\limsup_{N\to\infty} P^N = \tilde{C}\,. \tag{4.76}$$

Similar to the treatment of Lemma 4.9, the proof is split into two parts: First, convergence along a subsequence is shown, and thereafter, the convergence is extended to the whole sequence. Lemma 4.11 then immediately implies

$$\limsup_{N\to\infty}\frac{Z^{N,2}(T_{\epsilon N})}{K(T_{\epsilon N})} \leq \tilde{C}\,. \tag{4.77}$$

Proof (part 1: Convergence along the subsequence $N \mapsto N^2$). Using

$$\begin{aligned}\mathbb{E}\left[K^{(k)}(f(N,\ k))\right] &= \mathbb{E}\left[W + w^{(k)}(f(N,\ k))\right]e^{\alpha f(N,\ k)}\\ &= \mathbb{E}\left[W + w^{(k)}(f(N,\ k))\right]\frac{\eta\epsilon N}{A^{-1}k}\,,\end{aligned} \tag{4.78}$$



the expectation can be bounded by $\eta C\epsilon$, for any $\eta > 1$:

$$\begin{aligned}
\mathbb{E}\left[P^N\right] &\leq \frac{1}{\epsilon N}\sum_{k\in I^N}\mathbb{E}\left[1_{\{U_k\leq k/N\}}K^{(k)}(f(N,\ k))\right] \\
&\leq A\eta\sum_{k\in I^N}\mathbb{P}\left(U_k\leq\frac{k}{N}\right)\mathbb{E}\left[\left(W^{(k)}+w^{(k)}(f(N,\ k)\right)\frac{1}{k}\right] \\
&\leq A\eta\sup_{t\geq 0}\mathbb{E}\left[(W+w(t)\right]\sum_{k\in I^N}\frac{k}{N}\frac{1}{k} \\
&= A\eta A\frac{\lceil\epsilon N\rceil-\left\lceil N^{\frac{1}{5}}\right\rceil}{N}\sup_{t\geq 0}\mathbb{E}\left[(W+w(t)\right]\ . \qquad (4.79)
\end{aligned}$$

This gives the claim in expectation. Now recall that, if B is a set and Y a random variable, and 1_B is independent of Y, then

$$\mathbb{V}\text{ar}\left[1_B Y\right]=\mathbb{E}\left[1_B Y^2\right]-\mathbb{E}\left[1_B\right]^2\mathbb{E}\left[Y\right]^2\leq\mathbb{P}\left(B\right)\mathbb{V}\text{ar}\left[Y\right]\ . \qquad (4.80)$$

Hence, for fixed $\delta > 0$,

$$\begin{aligned}
\mathbb{P}\left(P^N-C\epsilon>\delta\right) &\leq \mathbb{P}\left(\left|P^N-\mathbb{E}\left[P^N\right]\right|^2>\delta^2\right) \\
&\leq \left(\frac{1}{\delta\epsilon N}\right)^2\sum_{k\in I^N}\mathbb{V}\text{ar}\Bigg[1_{\{U_k\leq k/N\}} \\
&\qquad \left(W^{(k)}+w^{(k)}(f(N,\ k)\right)\frac{\eta\epsilon N}{A^{-1}k}\Bigg] \\
&\leq \left(\frac{A^2\eta}{\delta}\right)^2\sum_{k\in I^N}\frac{k}{N}\frac{1}{k^2} \\
&\qquad \sup_{t\geq 0}\mathbb{E}\left[(W+w(t))^2\right]\ . \qquad (4.81)
\end{aligned}$$

This upper bound is of order $O(N^{-1}\log N)$, and thus summable along the subsequence $N\mapsto N^2$. □

Now we have to extend convergence to the whole sequence.

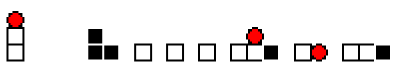
Proof (part 2: Convergence for the whole sequence). Let I^N be as in Lemma 4.11. Define

$$J(N^2) \ = \ I^{(N+1)^2}\setminus I^{N^2}\ . \qquad (4.82)$$

There are three possibilities for $M\mapsto P(M)$ to change: First, the new summands that are added atop increase the proportion, secondly, old summands may grow unexpectedly quickly or slowly; and thirdly, there may be a black fork that is cut away and becomes white; and additionally, in this case there is the possibility that further up on that fork another index gets coloured.

Only the first two possibilities actually increase $P(M)$, and we obtain thus for any fixed $\delta > 0$

$$\begin{aligned}
\mathbb{P}\left(\sup_{M\in J(N^2)}\left(P^M-P^{N^2}\right)>2\delta\right) &\leq \mathbb{P}\left(\sum_{k\in J(N^2)}1_{A_k}\frac{K^{(k)}(f((N+1)^2,\ k)}{\epsilon N^2}>\delta\right) \\
&\quad +\mathbb{P}\left(\sup_{M\in J(N^2)}\sum_{k\in I^{N^2}}1_{A_k}\Delta K(k,\ M)>\delta\right)\ , \\
&\qquad (4.83)
\end{aligned}$$



where we abbreviate

$$A_k = \{U_k \leq \frac{k}{N^2}\} \tag{4.84}$$

and

$$\Delta K(k,\ M) = \frac{K^{(k)}(f(M,\ k))}{\epsilon M} - \frac{K^{(k)}(f(N^2,\ k))}{\epsilon N^2}\,. \tag{4.85}$$

For the first summand, note that

$$\#J(N^2) = \frac{\epsilon}{A}(2N+1) + o(1)\,. \tag{4.86}$$

The expectation of the term in question is thus of order N^{-1}. Hence,

$$\begin{aligned}\mathbb{P}\left(\frac{1}{\epsilon N^2}\sum_{k\in J(N^2)} K^{(k)}(f((N+1)^2,\ k)) > \delta\right) \quad &\leq \quad \frac{\#J(N^2)}{\left(\epsilon N^2(\delta - O(N^{-1}))\right)^2} \\ &\qquad \cdot \sup_{t\geq 0}\mathbb{E}\left[(W+w(t))^2\right]\,,\end{aligned} \tag{4.87}$$

and this is summable. For the second summand, note that again the mean is of order $o(1)$; also,

$$\begin{aligned}A^{-1}k\cdot\Delta K(k,\ M) \quad &\leq \quad \frac{(N+1)^2 - N^2}{N^2}W^{(k)} + \frac{(N+1)^2}{N^2}w^{(k)}(f((N+1)^2,\ k)) \\ &\qquad -w^{(k)}(f(N^2,\ k))\,,\end{aligned} \tag{4.88}$$

uniformly in M. If we abbreviate the right hand side of (4.88) with $Y(N,\ k)$, we obtain

$$\begin{aligned}\mathbb{P}\left(\sup_{M\in J(N^2)}\sum_{k\in I^{N^2}}\Delta K(k,\ M) > \delta\right) \quad &\leq \quad \frac{1}{(\delta - o(1))^2}\sum_{k\in I(N^2)}\frac{k}{N^2}\frac{1}{A^2k^2} \\ &\qquad \cdot\mathbb{V}\mathrm{ar}\left[Y(N,\ k)\right]\,,\end{aligned} \tag{4.89}$$

and we can again bound the variance of $Y(N,\ k)$ by the second moment of $W + w(\cdot)$. The resulting expression is of order $N^{-2}\log N$ and thus summable. □

This finishes the proof of Theorem 4.1.

4.6 Extensions

In [DG2010], Dawson and Greven use arguments like the above to show, in a weaker coupling, the following:

Corollary 4.13. *Let the constant C be as in Theorem 4.1. The proportion of black colonies amongst black and white colonies at time $T_{\epsilon N}$ is asymptotically small in expectation:*

$$\limsup_{N\to\infty}\mathbb{E}\left[\frac{Z^N(T_{\epsilon N})}{K(T_{\epsilon N})}\right] \leq C\epsilon\,. \tag{4.90}$$

Proof. Apply Fatou's Inequality to the proportion (4.1). □

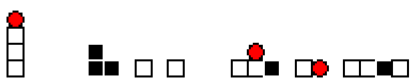
The Theorem does not say anything about the time $T^N_{\epsilon N}$; in principle, the trajectories of $K(t)$ and $K^N(t)$ could be torn apart in the time interval $[T_{\epsilon N},\ T^N_{\epsilon N}]$. The following corollary shows that this is not the case for small ϵ.

Corollary 4.14. *Assume that the constants C and ϵ are as in Theorem 4.1.*

1. *Assume that the equation*
$$\tilde{\epsilon}(1 - C\tilde{\epsilon}) = \epsilon \tag{4.91}$$
has its smallest solution $\tilde{\epsilon}$ in the interval $(0,\ 1)$. Then $K(t)$ can not get unboundedly large in comparison to $K^N(t)$ at time $T^N_{\epsilon N}$; more precisely,
$$\limsup_{N\to\infty} \frac{K(T^N_{\epsilon N})}{K^N(T^N_{\epsilon N})} \leq \frac{\tilde{\epsilon}}{\epsilon}\,. \tag{4.92}$$

2. *For large N,*
$$T_{\epsilon N} \leq T^N_{\epsilon N} \leq T_{\tilde{\epsilon} N}\,. \tag{4.93}$$
This implies for all $\eta > 0$ that ultimately
$$T^N_{\epsilon N} - \frac{1}{\alpha}\log\frac{\epsilon N}{W} \in [0 - \eta,\ \frac{1}{\alpha}\log\frac{\tilde{\epsilon}}{\epsilon} + \eta]\,. \tag{4.94}$$

Proof. Fix some $\delta > 0$ and let $\tilde{\epsilon}$ be the solution to
$$\tilde{\epsilon}(1 - C\tilde{\epsilon}) = \epsilon(1 + \delta)\,. \tag{4.95}$$
Theorem 4.1 implies that for large N
$$K(T^N_{\epsilon N}) < \tilde{\epsilon} N\,. \tag{4.96}$$
Otherwise, we had $T_{\tilde{\epsilon} N} \leq T^N_{\epsilon N}$. At time $T_{\tilde{\epsilon} N}$, we knew
$$K(T_{\tilde{\epsilon} N}) - K^N(T_{\tilde{\epsilon} N}) \leq \tilde{\epsilon}^2 C N\,, \tag{4.97}$$
which implied for infinitely many N
$$\lceil \epsilon N \rceil = K^N(T^N_{\epsilon N}) \geq K^N(T_{\tilde{\epsilon} N}) \geq \lceil \tilde{\epsilon} N \rceil - \tilde{\epsilon}^2 C N\,. \tag{4.98}$$
This is a contradiction because of the choice of $\tilde{\epsilon}$. Letting $\delta \to 0$ proves the corollary. □



5 Deterministic evolution on large scales

Chapter 4 showed that the trajectories of $K(t)$ and $K^N(t)$ do not differ much up to time $T^N_{\epsilon N}$. Corollary 2.11 showed that, far to the right of $t = 0$, $K(t)$ evolves deterministically in the sense that

$$T_{\epsilon N} - T_{\log N} = \frac{1}{\alpha} \log \epsilon N - \frac{1}{\alpha} \log \log N + o(1)\,, \qquad (N \to \infty)\,. \tag{5.1}$$

The next step is to ask if $K^N(t)$ shares this property. The goal of this chapter is thus to find an analogue of (5.1) for the N Colony System. We will not be able to find an almost sure limit assertion, but at least to show that the variance of $T^N_{\epsilon N} - T^N_{\log N}$ vanishes in some weaker conditional sense. Note that the branching process arguments as used in Corollary 2.11 do not allow to conclude L^2 convergence in (5.1).

The idea is to find a random walk

$$S(t) \tag{5.2}$$

that evolves below $K^N(t)$, such that the variance of the corresponding hitting time

$$T^S_{\epsilon N} - T^S_{\log N} \tag{5.3}$$

of the random walk goes to zero for $N \to \infty$. The difficult part then is to relate this variance with the variance of $T^N_{\epsilon N} - T^N_{\log N}$. In order to ensure that the random walk remains underneath, the strategy will be to let this walk do more steps downwards; it seems thus immediately clear that the walk "collects more randomness" which then mirrors in its hitting time. However, we are only able to make this rigorous for conditional variance, where we condition on the starting point $\Psi^N(T^N_{\log N})$.

5.1 Statement of results

First, we introduce the approximating random walk where we do not specify the coupling to the process $K^N(t)$ yet.

Definition 5.1 (The harmonic random walk)**.**
Let S be a continuous time Markov chain with state space $\mathbb{N}$ satisfying the following conditions:

1. *The process starts in state 1, i. e. $S(0) = 1$ almost surely.*
2. *There exists a constant $\rho \in (0, 1)$ such that the process stays in state i an exponentially distributed time with parameter $c\rho i$, $i \in \mathbb{N}$.*
3. *The process steps from state 1 with probability 1 to state 2.*
4. *There exists a constant $p \in [0, \frac{1}{2})$ such that the process steps from state $i \in \mathbb{N} \setminus \{1\}$ to state $i - 1$ with probability p and to state $i + 1$ with probability $1 - p$.*

The process then is called a harmonic random walk*. Let for such a process S the quantity*

$$T^S_M = \inf\{t \geq 0 \;:\; S(t) \geq M\} \tag{5.4}$$

denote the hitting time of S at level M, $M \in \mathbb{R}_+$.

The assertion is that the constants can be chosen such that the walk reaches level $\lceil \epsilon N \rceil$ always later than the functional $K^N(t)$. This is confirmed by simulations as shown in Figure 5.1 and stated rigorously in Theorem 5.2 below.

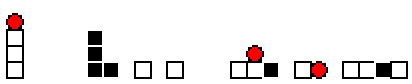


Figure 5.1: Comparison of $K^N(t)$ (black) with a harmonic random walk $S(t)$ (red). The parameters are $N = 1000$, $s = 3$, $c = 1$, $d = 0.5$. The random walk steps to the left with probability $p = 0.4$. The graphic also shows that the attribute *harmonic* is misleading: If the waiting times were i. i. d., the walk would move to the right with linear speed. Since the waiting times decrease, the red curve grows super-linearly (in fact, exponentially). We will exploit below that initially the red curve evolves below the black.

Theorem 5.2 (Variances of hitting times)**.**
There exist some ϵ, $\rho \in (0, 1)$ and $p \in [0, \frac{1}{2})$ such that the following hold:

1. *For fixed N, the N Colony System ζ^N and the harmonic random walk $S \equiv S(p, \rho)$ can be defined on one probability space such that almost surely*
$$T^N_{\epsilon N} - T^N_{\log N} \leq T^S_{\epsilon N} - T^S_{\log N} \,. \tag{5.5}$$

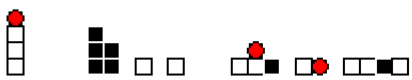
2. *The variance of the hitting time of S is an upper bound for the conditional variance of the N Colony System:*
$$\begin{aligned} \sup_\psi \mathbb{V}ar\left[T_{\epsilon N} - T_{\log N} \mid \Psi(T_{\log N}) = \psi\right] &\leq \mathbb{V}ar\left[T^S_{\epsilon N} - T^S_{\log N}\right] \,, \\ \sup_\psi \mathbb{V}ar\left[T^N_{\epsilon N} - T^N_{\log N} \mid \Psi^N(T^N_{\log N}) = \psi\right] &\leq \mathbb{V}ar\left[T^S_{\epsilon N} - T^S_{\log N}\right] \,. \end{aligned} \tag{5.6}$$
Here, the supremum is taken over all states $\psi \in \mathcal{M}_{fin}(\mathbb{N})$ with $\psi_i \in \mathbb{N}_0$ $(i \in \mathbb{N})$ and
$$\sum_{k \geq 1} \psi(k) = \lceil \log N \rceil \,. \tag{5.7}$$

3. *Expectation and variance of $T^S_{\epsilon N} - T^S_{\log N}$ can for $N \to \infty$ be bounded as follows:*
$$\begin{aligned} \mathbb{E}\left[T^S_{\epsilon N} - T^S_{\log N}\right] &\leq \frac{1}{\rho c} \sum_{k = \lceil \log N \rceil}^{\lceil \epsilon N \rceil - 1} \frac{1}{k} = \frac{1}{\rho c} \log\left(\frac{\epsilon N}{\log N}\right) + o(1) \,. \\ \mathbb{V}ar\left[T^S_{\epsilon N} - T^S_{\log N}\right] &\leq \frac{2}{(\rho c)^2} \sum_{k = \lceil \log N \rceil}^{\lceil \epsilon N \rceil - 1} \frac{1}{k^2} = \frac{2}{(\rho c)^2} \left(\frac{1}{\log N} - \frac{1}{\epsilon N}\right) + o(1) \,. \end{aligned} \tag{5.8}$$

Remark 5.3.

1. *The left hand sides of the inequalities* (5.6) *is at least formally well-defined because any ψ satisfying* (5.7) *is reachable within a finite number of steps; any such path has non-zero probability.*

2. *We do not claim here that the harmonic random walk S always stays below K^N. However, we construct S as a time change of a certain process K^S that is driven by a subset of particles of ζ^N. The process K^S does not have exponential waiting times but stays always below K^N.*



3. *Modifying the argument,* (5.5) *can also be shown uniformly in* N.
4. *Using the usual conditional variance formula, the restriction to the conditional variance would be unnecessary if one knew*

$$\lim_{N\to\infty} \mathbb{V}ar\left[\mathbb{E}\left[T^N_{\epsilon N} - T^N_{\log N} \mid \Psi^N(T^N_{\log N})\right]\right] = 0\,. \tag{5.9}$$

This expression can be paraphrased as "how much does the expected hitting time vary when one modifies the initial condition?"; and the message of the proof below is that the expected hitting time does not at all depend on the initial state. The difficulty is to formalize this intuition.

5. *The removal of the conditioning would imply that there exists some deterministic function* $f(N)$ *such that*

$$T^N_{\epsilon N} - T^N_{\log N} - f(N) \to 0 \tag{5.10}$$

in probability. This in turn would imply that $T^N_{\epsilon N} - T_{\epsilon N}$ *is asymptotically nonrandom, as can be seen via the representation*

$$T^N_{\epsilon N} - T_{\epsilon N} = \left(T^N_{\epsilon N} - T^N_{\log N}\right) + \left(T^N_{\log N} - T_{\log N}\right) + \left(T_{\log N} - T_{\epsilon N}\right)\,. \tag{5.11}$$

Here, the second summand converges in probability to zero, because the collision probability up to time $T_{\log N}$ *is* $o(N)$. *The third converges to some constant by virtue of the CMJ Theory (cf. Corollary 2.11).*

In Section 5.2, we prove the first assertion; this is the only one that really is specific to the N Colony System. The remainder of the proof relies on certain recurrence equations for the mean and variance of hitting times; these are presented and motivated in Section 5.3. We then turn to the third assertion which is shown in a self-contained calculation in Section 5.4. Finally, in a technically involved argument that is described in Section 5.5 and carried out in Appendix B.3, the recurrence equations that are associated with the N Colony System are reduced to those of the random walk. This is used to prove the second assertion. Since the arguments immediately extend to the Collision Free System, we only consider the N Colony System.

In order to abbreviate expressions like (5.7) in the future, we make the following definition.

Definition 5.4 (Partition of the state space)**.**
Consider both the N *Colony System and the Collision Free System. For* $i \in \mathbb{N}$, *define*

$$[i] = \{\psi \in \mathcal{M}_{fin}(\mathbb{N}) \ : \ \psi(k) \in \mathbb{N}_0 \textit{ for all } k \textit{ and } \sum_{k\geq 1} \psi(k) = i\}\,. \tag{5.12}$$

The set $[i]$ *contains all configurations that resemble* i *populated colonies. We refer to* $[i]$ *as a* macrostate *and to any* $\psi \in [i]$ *as a* microstate *attached to the macrostate* $[i]$. *A special representative of the set* $[i]$ *is*

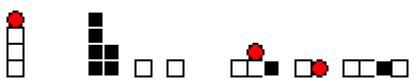
$$\psi_i = i \cdot \delta_1\,. \tag{5.13}$$

This is the configuration with exactly one particle per colony and is called the minimal configuration of $[i]$.

The idea of the reduction of the N Colony System to the random walk (which happens in the final step below) is that the dynamics depend primarily on the macrostates such that changes in the microstates can mostly be ignored.

Remark 5.5. *In the case of the Collision Free System, the heuristic argument to obtain* (5.8) *is the following: In state* $K_t = i$, *there are* $O(i)$ *particles, each one carrying an exponential migration clock. The time between migrations has thus mean* $O(i^{-1})$ *and Variance* $O(i^{-2})$. *This leads to the following calculation:*

$$\mathbb{V}ar(T_{\epsilon N} - T_{\log N}) = \sum_{k=\lceil \log N\rceil}^{\lceil \epsilon N\rceil - 1} \mathbb{V}ar(T_{i+1} - T_i) = O\left(\sum_{k=\lceil \log N\rceil}^{\lceil \epsilon N\rceil - 1} \frac{1}{i^2}\right)\,. \tag{5.14}$$



The right hand side is bounded by the remainder term of a convergent sum and thus vanishes for $N \to \infty$. The problem that arises here is that the one step hitting times $T_{i+1} - T_i$ depend on the internal states $\Psi(t)$ and are thus not independent. The intuition is that these dependencies are rather weak and quickly decaying in time (this is supported by the fact that $\Psi(t)K(t)^{-1}$ converges to some Ψ_∞ when $t \to \infty$); but we are not able to make the calculation rigorous.

5.2 Proof of the first assertion: Coupling to the random walk

The idea is to take a certain subset of particles of ζ^N and let the number of colonies inhabited by these particles perform a random walk. This random walk K^S is defined in Definition 5.6. Lemma 5.7 and Lemma 5.9 show that this process is a harmonic random walk up to a random time change; Corollary 5.10 then defines S as this time change of K^S and summarizes that it satisfies both Definition 5.1 and inequality (5.5).

Definition 5.6 (The embedded random walk of the N Colony System)**.**
Fix some $\epsilon \in (0, 1)$. The system ζ^S with state space $\mathbb{S}$ evolves as follows:

- *The system starts in the configuration with one particle on the first colony.*
- *Between migration steps, the system follows the same rules as the N Colony System or the Collision Free System, i. e. particles give birth at rate s and pairs at the same location coalesce at rate d.*
- *Particles migrate as before with rate c. Any migration collides with fixed probability ϵ. The particle is then placed on some arbitrarily chosen inhabited colony. With probability $1 - \epsilon$, it is placed on a free colony.*
- *Finally, after each migration step, all particles except for one randomly selected particle per colony are removed from the system.*

Migrations are thus transitions from some $\psi \in [i]$ to $\phi \in \{\psi_{i-1}, \psi_i, \psi_{i+1}\}$, where these state are the minimal configurations as introduced in Definition 5.4. Let

$$K^S(t) \tag{5.15}$$

denote the number of inhabited colonies in system ζ^S at time t. Define for $i \in \mathbb{N}$

$$\begin{aligned} \tilde{T}_i^S &= \inf\{t \geq 0 \ : \ K^S(t) = i\}, \\ \tilde{U}_i^S &= \inf\{t \geq \tilde{T}_i^S \ : \ K^S(t) \neq i\} \end{aligned} \tag{5.16}$$

the hitting and first leaving times at level i.

Now define the N Colony System ζ^N on the same probability space such that, from time $T^N_{\log N}$ and $T^S_{\log N}$ respectively onwards, all particles in system ζ^S are identified with a subset of particles of ζ^N.

In order to associate the process K^S with Definition 5.1, we have to control both the moments of the waiting times and the jump probabilities of the embedded jump chain. We consider the holding times first. Recall that p_0 is the probability that the very first migrant is not single (cf. Proposition 2.14).

Lemma 5.7. *Let K^S and $\tilde{T}^S_\cdot$, $\tilde{U}^S_\cdot$ be as in Definition 5.6.*

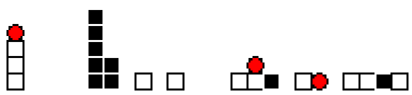
1. *There exists a constant $\rho \in (0, 1)$ and an independent array $\{\gamma_i^{(n)}\}_{n,i\in\mathbb{N}}$ such that, for all $i \in \mathbb{N}$,*

$$\gamma_i^{(n)} \stackrel{d}{=} Exp(\rho c i) \tag{5.17}$$

and

$$\tilde{T}_i^S - \tilde{U}_i^S \leq \gamma_i^{(1)} \tag{5.18}$$

almost surely.



2. *Let* $\tilde{T}^S_{i,n}$, $\tilde{U}^S_{i,n}$ *denote the corresponding quantities for the* n^{th} *visit at state* n. *Then, on the set where* K^S *visits state* i *at least* n *times,*

$$T^S_{i,n} - U^S_{i,n} \leq \gamma_i^{(n)} \,. \tag{5.19}$$

3. *A possible choice for* ρ *is*

$$\rho = \epsilon(1 - p_0) + (1 - \epsilon)p_0 \,. \tag{5.20}$$

Proof. The probability that the first migrant collides is ϵ; the probability that it is single is p_0. The migration leads to a configuration $\phi \notin [i]$ in the following cases: It is a colliding single particle (which leads to a configuration $\phi \in [i-1]$) or it is a non-colliding non-single particle (which leads to $\phi \in [i+1]$). By independence, any of these happens with probability

$$\epsilon(1 - p_0) + (1 - \epsilon)p_0 \,. \tag{5.21}$$

After each migration that does not change the macrostate $[i]$ the system falls back into its minimal configuration ψ_i. Hence, the number of migrations until the first change in the macrostate occurs is geometrically distributed with parameter ρ, where ρ is given by expression (5.21).

The time between migrations is bounded from above by the time that the i initially fixed particles need, which is $\mathrm{Exp}(ci)$ distributed. Hence, we obtain in self explaining notation

$$\tilde{U}^S_i - \tilde{T}^S_i \leq \sum_{k=1}^{\mathrm{Geom}(\rho)} \mathrm{Exp}_k(ci) \,, \tag{5.22}$$

where all quantities on the right are independent. We can now apply Lemma 3.2 of [SA2003] (cf. the following remark) in order to obtain that the right hand side of (5.22) is exponentially distributed with parameter ρci. □

Remark 5.8. *We used that*

$$\sum_{k=1}^{Geom(\rho)} Exp_k(ci) \stackrel{d}{=} Exp(\rho ci) \,, \tag{5.23}$$

when the quantities on the left hand side are independent. Asmussen uses in Lemma 3.2 of [SA2003] a nice coupling argument to show this: Consider a continuous time Markov Chain on the states

$$\{A,\, B,\, C\} \tag{5.24}$$

that starts in state A, *is symmetric in states* A *and* B *and terminates in state* C. *The assertion is that, when the transition rates are correctly chosen, both sides of* (5.23) *give the hitting time at state* C*:*

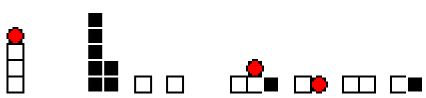
- *The transition rate from* A *to* C *as well as from* B *to* C *is defined as* ρci. *By the memoryless property, this establishes the right hand side of* (5.23).
- *The transition rate from* A *to* B *and vice versa is defined as* $(1-\rho)ci$. *The probability to jump from* A *to* C *instead from* A *to* B *is then*

$$\frac{\rho ci}{(1-\rho)ci + \rho ci} = \rho \,; \tag{5.25}$$

this establishes the geometric sum on the left hand side of (5.23). *The assertion is proven by noting that the total rate to leave state* A *is given by*

$$\rho ci + (1-\rho)ci = ci \,, \tag{5.26}$$

which establishes the summands of the geometric sum.



We now show that the embedded jump chain is a random walk that is more likely to move upwards than downwards.

Lemma 5.9. *If $\epsilon < p_0$, then*

$$\sup_{N\in\mathbb{N}} \sup_{\lceil \log N\rceil \leq i < \lceil \epsilon N\rceil} \sup_{n\in\mathbb{N}} \mathbb{P}(K^S(\tilde{U}^S_{i,n}) = i-1) < \frac{1}{2}\,. \tag{5.27}$$

Here, we define $K^S(\tilde{U}^S_{i,n}) = \infty$ when $\tilde{U}^S_{i,n} = \infty$. Conditioned on the set where K^S visits state i at least n times, these probabilities do not depend on i or n, if $i \in \mathbb{N}\setminus\{1\}$.

Proof. Fix some $i \geq 2$. In the proof of Lemma 5.7, the parameter ρ was identified as the probability that a migration changes the macrostate of K^S. We now ask for the probability that a migration leads to a decrease of K^S instead of an increase.

There are two possibilities for K^S to change: A colliding single particle migration decreases K^S; this happens with probability $\epsilon(1-p_0)$. A non-colliding non-single particle migration increases K^S; this happens with probability $(1-\epsilon)p_0$. By construction, these probabilities do not depend on i nor on the number of migrations that happened previously. Abbreviate

$$\begin{aligned} M^-(k) &= \left\{ \begin{array}{c} \text{a colliding single particle migration} \\ \text{occurs in the } k^{th} \text{ migration} \end{array} \right\}, \\ M^+(k) &= \left\{ \begin{array}{c} \text{a noncolliding nonsingle particle migration} \\ \text{occurs in the } k^{th} \text{ migration} \end{array} \right\}. \end{aligned} \tag{5.28}$$

Then, using that the migrations are independent,

$$\begin{aligned} \mathbb{P}(K^S(\tilde{U}^S_i) = i-1) &= \sum_{k\geq 1} \mathbb{P}\left(M^-(k) \cap \bigcap_{m=1}^{k-1} (M^-(m)\cup M^+(m))^c \right) \\ &= \sum_{k\geq 1} \epsilon(1-p_0)(1-\rho)^{k-1} \\ &= \frac{\epsilon(1-p_0)}{\epsilon(1-2p_0)+p_0}\,. \end{aligned} \tag{5.29}$$

This expression is monotonically increasing in ϵ and equals $\frac{1}{2}$ for $\epsilon = p_0$. □

We conclude this section with the following corollary.

Corollary 5.10. *There exists a random time change τ such that the process*

$$S(t) = K^S(\tau(t)) \tag{5.30}$$

is a harmonic random walk as introduced in Definition 5.1. Moreover, the hitting time

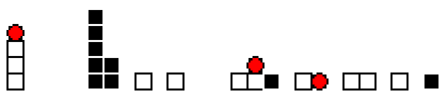
$$T^S_\cdot = \inf\{t \geq 0 \;:\; S(t) \geq \cdot\} \tag{5.31}$$

satisfies

$$T^N_{\epsilon N} - T^N_{\log N} \leq T^S_{\epsilon N} - T^S_{\log N} \tag{5.32}$$

almost surely.

Proof. Dilate in system ζ^S the time between migrations to the time that it takes for the exponential clocks that are attached to the i initially selected particles to ring. The claim then follows from the preceding two Lemmas. □



5.3 The tool for the proof of the remaining assertions: Recurrence equations

This chapter aims to show in an introductory example the recurrence equations at work. We consider a random walk in discrete time; the general procedure in continuous time is then summarized in Proposition 5.12. This result is then applied to the harmonic random walk in Section 5.4.

Proposition 5.11. *Consider an asymmetric random walk on* $\mathbb{Z}$ *that starts in* 0 *and makes steps of size* 1 *to the left with fixed probability* $p < \frac{1}{2}$ *and to the right with probability* $q = 1-p$. *The hitting time* τ_k *when level* $k \in \mathbb{N}$ *is reached by the random walk satisfies*

$$\begin{aligned} \mathbb{E}\left[\tau_k\right] &= k\frac{1}{1-2p}\,, \\ \mathbb{V}ar\left[\tau_k\right] &= k\frac{4p(1-p)}{(1-2p)^3}\,. \end{aligned} \tag{5.33}$$

Proof. The relation

$$\tau_k = \sum_{m=1}^{k} \left(\tau_m - \tau_{m-1}\right) \tag{5.34}$$

leads to

$$\begin{aligned} \mathbb{E}\left[\tau_1\right] &= q + p(1 + 2\mathbb{E}\left[\tau_1\right])\,, \\ \mathbb{E}\left[\tau_k\right] &= k\mathbb{E}\left[\tau_1\right]\,, \end{aligned} \tag{5.35}$$

and this implies the first assertion. Similarly, we have

$$\begin{aligned} \mathbb{E}\left[\tau_1^2\right] &= q + p\mathbb{E}\left[(1+\tau_2)^2\right] \\ &= 1 + 4p\mathbb{E}\left[\tau_1\right] + p\mathbb{E}\left[(\tau_2 - \tau_1 + \tau_1)^2\right] \\ &= 1 + 4p\mathbb{E}\left[\tau_1\right] + p\mathbb{E}\left[(\tau_2 - \tau_1)^2 + 2(\tau_2-\tau_1)\tau_1 + (\tau_1)^2\right] \\ &= 1 + 4p\mathbb{E}\left[\tau_1\right] + 2p\mathbb{E}\left[\tau_1^2\right] + 2p\mathbb{E}\left[\tau_1\right]^2\,. \end{aligned} \tag{5.36}$$

This yields

$$\mathbb{E}\left[\tau_1^2\right] = \frac{1+2p(1-2p)}{(1-2p)^3} \tag{5.37}$$

and

$$\mathbb{V}\mathrm{ar}\left[\tau_1\right] = \frac{4p(1-p)}{(1-2p)^3}\,. \tag{5.38}$$

Finally,

$$\mathbb{V}\mathrm{ar}\left[\tau_k\right] = \sum_{m=1}^{k} \mathbb{V}\mathrm{ar}\left[\tau_m - \tau_{m-1}\right] = k\frac{4p(1-p)}{(1-2p)^3}\,. \tag{5.39}$$

□

The strategy was to condition on the first step of the random walk, and to start the dynamics from there anew. The generalization of this to continuous time processes reads as follows.

Proposition 5.12. *Let* $(X_t)_{t\geq 0}$ *be a nonexplosive continuous time Markov chain with countable state space* $\mathbb{T}$ *and* Q *matrix* $(q(s,\,t))_{s,t\in\mathbb{T}}$ *with the usual convention that*

$$q(s,\,s) = -\sum_{\substack{t\,\in\,\mathbb{T}\,,\\ t\,\neq\,s}} q(s,\,t)\,. \tag{5.40}$$

Assume that, under the law $\mathbb{P}_s$, the process starts in $s \in \mathbb{T}$. For $H \subset \mathbb{T}$ consider the hitting time

$$\tau_H = \inf\{t \geq 0 \ : \ X_t \in H\}\,. \tag{5.41}$$

Assume that

$$|q(s,\, s)| > 0 \quad \text{for all } s \notin H\,. \tag{5.42}$$

Abbreviate expectation and variance of τ_H with

$$\begin{aligned} e(s) &= \mathbb{E}_s\left[\tau_H\right]\,, \\ v(s) &= \mathbb{V}ar_s\left[\tau_H\right]\,. \end{aligned} \tag{5.43}$$

Then, the vector $(e(s))_{s\in\mathbb{T}}$ is the minimal non-negative solution to the following system of equations:

$$\begin{aligned} -\sum_{t\in\mathbb{S}} q(s,\, t)e(t) &= 1\ ,\ \text{if } s \notin H \\ e(s) &= 0\ ,\ \text{if } s \in H\,. \end{aligned} \tag{5.44}$$

Similarly, $(v(s))_{s\in\mathbb{T}}$ satisfies

$$\begin{aligned} \sum_{\substack{t\,\in\,\mathbb{T}\,,\\ t\,\neq\, s}} \frac{q(s,\, t)}{|q(s,\, s)|}v(t) + f(s) &= v(s)\ ,\ \text{if } s \notin H \\ 0 &= v(s)\ ,\ \text{if } s \in H\,, \end{aligned} \tag{5.45}$$

where

$$f(s) = \sum_{\substack{t\,\in\,\mathbb{T}\,,\\ t\,\neq\, s}} \frac{q(s,\, t)}{|q(s,\, s)|}\mathbb{E}\left[\left(e(t) - e(s) + \gamma_s^{(0)}\right)^2\right]\,. \tag{5.46}$$

Here, $\gamma_s^{(0)}$ is exponentially distributed with parameter $|q(s,\, s)|$. For the variance, no minimality assertion is made.

For the ease of computation, we state the linear systems that we have to solve in a different way.

Corollary 5.13. *The vectors $(e(s))_{s\in\mathbb{T}}$, $(v(s))_{s\in\mathbb{T}}$ solve the following systems of equations:*

$$\begin{aligned} \hat{Q}e &= b\,, \\ \hat{Q}v &= f\,. \end{aligned} \tag{5.47}$$

Here, the vector b is given via

$$b(s) = \frac{1}{|q(s,\, s)|} \qquad (s \in \mathbb{T}) \tag{5.48}$$

and the entries of the matrix $\hat{Q}$ are obtained from the Q-matrix as follows:

$$\hat{q}(s,\, t) = -\frac{q(s,\, t)}{|q(s,\, s)|} \qquad (s \in \mathbb{T}\setminus H\,,\ t \in \mathbb{T})\,. \tag{5.49}$$

Proof of Proposition 5.12. We combine Theorems 1.3.5 and 3.3.3 of [JN1997] (where formulas for the expectation are given, and minimality is shown) and extend Lemma 7 of [IG2007] (where a formula for the variance in discrete time is given).

Let

$$(\tilde{X}(m))_{m\in\mathbb{N}_0} \tag{5.50}$$

be the embedded jump chain. Let

$$(\gamma_t^{(n)})_{n\in\mathbb{N}_0,\, t\in\mathbb{T}} \tag{5.51}$$

be an array of independent holding times such that, for any n, $\gamma_t^{(n)}$ is exponentially distributed with parameter $|q(t,\, t)|$. We now work with the usual holding time construction. This implies

$$\tau_H = \sum_{k=0}^{\tilde{\tau}_H - 1} \gamma_{\tilde{X}(k)}^{(k)} \quad \text{and} \quad \mathbb{E}_s\left[\tau_H - \gamma_{\tilde{X}(0)}^{(0)} \mid \tilde{X}(1) = t\right] = \mathbb{E}_t\left[\tau_H\right]. \tag{5.52}$$

Hence,

$$\begin{aligned} e(s) &= \sum_{\substack{t\in\mathbb{T},\\ t\neq s}} \mathbb{E}_s\left[\gamma_s^{(0)} + \tau_H - \gamma_s^{(0)} \mid \tilde{X}(1) = t\right] \mathbb{P}_s\left(\tilde{X}(1) = t\right) \\ &= \sum_{\substack{t\in\mathbb{T},\\ t\neq s}} \left(\mathbb{E}_s\left[\gamma_s^{(0)}\right] + \mathbb{E}_t\left[\tau_H\right]\right) \mathbb{P}_s\left(\tilde{X}(1) = t\right) \\ &= \frac{1}{|q(s,\, s)|} + \sum_{\substack{t\in\mathbb{T},\\ t\neq s}} \frac{q(s,\, t)}{|q(s,\, s)|} e(t)\,. \end{aligned} \tag{5.53}$$

Rearranging terms yields (5.44).

For the variance, we expand

$$\begin{aligned} (\tau_H - e(s))^2 &= (\tau_H - \gamma_s^{(0)})^2 + 2\gamma_s^{(0)}(\tau_H - \gamma_s^{(0)}) + \left(\gamma_s^{(0)}\right)^2 \\ &\quad -2e(s)(\tau_H - \gamma_s^{(0)} + \gamma_s^{(0)}) + e(s)^2 \end{aligned} \tag{5.54}$$

and use the following two identities:

$$\mathbb{E}_t\left[\tau_H^2\right] = v(t) + e(t)^2 \tag{5.55}$$

and

$$\mathbb{E}_s\left[\gamma_s^{(0)}(\tau_H - \gamma_s^{(0)}) \mid \tilde{X}(1) = t\right] = \frac{e(t)}{|q(s,\, s)|}\,. \tag{5.56}$$

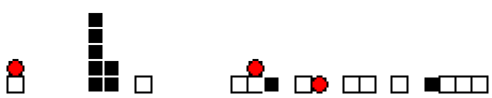
This leads to

$$\begin{aligned} \mathbb{E}_s\left[(\tau_H - e(s))^2 \mid \tilde{X}(1) = t\right] &= \left(v(t) + e(t)^2\right) + \frac{2e(t)}{|q(s,\, s)|} + 2\left(\frac{1}{|q(s,\, s)|}\right)^2 \\ &\quad -2e(s)\left[e(t) + \frac{1}{|q(s,\, s)|}\right] + e(s)^2 \\ &= v(t) + 2\frac{1}{|q(s,\, s)|}\left[e(t) - e(s)\right] \\ &\quad +2\left(\frac{1}{|q(s,\, s)|}\right)^2 + (e(t) - e(s))^2\,. \end{aligned} \tag{5.57}$$

Summing over t yields (5.46).

Minimality of e can be seen as follows (we copy the argument from [JN1997]): Let d be some non-negative solution to (5.44). Then, for $s \notin H$,

$$\begin{aligned} d(s) &= \frac{1}{|q(s,\, s)|} + \sum_{t\notin H} \frac{q(s,\, t)}{|q(s,\, s)|}\left[\frac{1}{|q(t,\, t)|} + \sum_{u\notin H} \frac{q(t,\, u)}{|q(u,\, u)|} d(u)\right] \\ &= \mathbb{E}_s\left[\gamma_{\tilde{X}(0)}^{(0)}\right] + \mathbb{E}_s\left[\gamma_{\tilde{X}(1)}^{(1)} 1_{\{\tilde{\tau}_H \geq 2\}}\right] + \sum_{t\notin H}\sum_{u\notin H} \pi(s,\, t)\pi(t,\, u) d(u)\,. \end{aligned} \tag{5.58}$$



Iterating this and using that the remainder term to the right is always nonnegative, this gives for all $n \in \mathbb{N}$

$$d(s) \geq \mathbb{E}_s \left[\sum_{k=0}^{(n \wedge \tilde{\tau}_H)-1} \gamma^{(k)}_{\tilde{X}(k)} \right] . \tag{5.59}$$

Monotone convergence yields

$$d(s) \geq \mathbb{E}_s \left[\sum_{k=0}^{\tilde{\tau}_H - 1} \gamma^{(k)}_{\tilde{X}(k)} \right] = \mathbb{E}_s \left[\tau_H \right] = e(s) . \tag{5.60}$$

□

Remark 5.14. *For Conditional Variance, the following variance formula holds:*

$$\mathbb{V}ar_s \left[\tau_H \right] = \mathbb{E}_s \left[\mathbb{V}ar_s \left[\tau_H \mid \tilde{X}(1) \right] \right] + \mathbb{V}ar_s \left[\mathbb{E}_s \left[\tau_H \mid \tilde{X}(1) \right] \right] . \tag{5.61}$$

Here, $\tilde{X}(1)$ denotes the position of the embedded jump chain after its first jump. Expression (5.46) can - up to the variance of the first holding time - be interpreted as the second summand. To see this, calculate the first summand as follows:

$$\begin{aligned}
\mathbb{E}_s \left[\mathbb{V}ar_s \left[\tau_H \mid \tilde{X}(1) \right] \right] &= \sum_{\substack{t \in \mathbb{T}, \\ t \neq s}} \frac{q(s,\, t)}{|q(s,\, s)|} \mathbb{V}ar_s \left[\tau_H \mid \tilde{X}(1) = t \right] \\
&= \sum_{\substack{t \in \mathbb{T}, \\ t \neq s}} \frac{q(s,\, t)}{|q(s,\, s)|} \left(\mathbb{V}ar_s \left[\tau_H - \gamma^{(0)}_s \mid \tilde{X}(1) = t \right] + \mathbb{V}ar_s \left[\gamma^{(0)}_s \right] \right) \\
&= \sum_{\substack{t \in \mathbb{T}, \\ t \neq s}} \frac{q(s,\, t)}{|q(s,\, s)|} v(t) + \mathbb{V}ar_s \left[\gamma^{(0)}_s \right] . \qquad (5.62)
\end{aligned}$$

5.4 Proof of the third assertion: Moments of the hitting time of the random walk

We now prove the third assertion of Theorem 5.2. The following treatment of the hitting times of the harmonic random walk is self-contained except for that it makes use of Proposition 5.12.

Proposition 5.15. *Let some constants $p \in [0,\, \frac{1}{2})$ and $c,\, \rho > 0$ be given. Let $(S(t))_{t \geq 0}$ be a harmonic random walk as defined in Definition 5.1. Then, for $i < n$ and $i \to \infty$,*

$$\begin{aligned}
\mathbb{E} \left[T^S_n - T^S_i \right] &= \frac{c\rho}{1 - 2p} \log \left(\frac{n}{i} \right) + o(1) , \\
\mathbb{V}ar \left[T^S_n - T^S_i \right] &= O(\frac{1}{i} - \frac{1}{n}) = o(1) . \qquad (5.63)
\end{aligned}$$

We apply Proposition 5.12 and consider the walk that starts in state i. According to Corollary 5.13, we have to solve the systems

$$\hat{Q} e = b \tag{5.64}$$

where b contains the expected waiting times in the various states. More precisely, the task is to solve

$$\begin{pmatrix} 1 & -1 & & & & \\ -p & 1 & -q & & & \\ & -p & 1 & -q & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p & 1 & -q \\ & & & & -p & 1 \end{pmatrix} \begin{pmatrix} e(1) \\ e(2) \\ e(3) \\ \vdots \\ e(n-2) \\ e(n-1) \end{pmatrix} = \rho c \begin{pmatrix} 1 \\ 2^{-1} \\ 3^{-1} \\ \vdots \\ (n-2)^{-1} \\ (n-1)^{-1} \end{pmatrix}. \tag{5.65}$$

We state the inverse of this matrix and refer the reader to Appendix B.2 for the proof.

Lemma 5.16. *Let $p \in [0,\ 1)$ and consider the $n \times n$ matrix $A^{(n)}$ with entries*

$$A^{(n)}_{11} = 1,\ A^{(n)}_{12} = -1,\ A^{(n)}_{1j} = 0,\ \text{if } j \geq 3\,, \tag{5.66}$$

and, for $i \geq 2$,

$$A^{(n)}_{ij} = \begin{cases} -p & , \text{ if } j = i-1 \\ 1 & , \text{ if } j = i \\ -(1-p) & , \text{ if } j = i+1 \\ 0 & , \text{ else}\,. \end{cases} \tag{5.67}$$

Define $q = 1 - p$ and

$$r = \frac{p}{q}\,. \tag{5.68}$$

Then, the inverse of $A^{(n)}$ is given by the product $R^{(n)}S^{(n)}$, where the factors have the following entries:

$$\begin{aligned} R^{(n)}_{ij} &= \sum_{k=(i-j)\vee 0}^{n-j} r^k\,, \\ S^{(n)}_{ij} &= \begin{cases} 1 & , \text{ if } i = j = 1 \\ \frac{1}{q} & , \text{ if } i = j > 1 \\ 0 & , \text{ else.} \end{cases} \end{aligned} \tag{5.69}$$

Proof. Cf. Appendix B.2. □

Remark 5.17. *In the case $p = 0$ (which corresponds to the Collision Free System), the formulas reduces to*

$$((A^{(n)})^{-1})_{i,j} = 1_{\{i \leq j\}} \tag{5.70}$$

which would make the analysis particularly simple. We do not treat this case separately since it is contained in the general case $p \in [0,\ \frac{1}{2})$.

We are now ready to proof Proposition 5.15. We consider the following simple analytic assertion in advance.

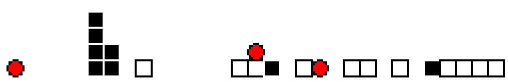
Lemma 5.18. *For $r \in [0,\ 1)$, the quantity*

$$g(n) = \sum_{i=1}^{n} \frac{r^{n-i}}{i} \tag{5.71}$$

satisfies

$$g(n) = O(n^{-1}) \tag{5.72}$$

for $n \to \infty$.



Proof. Fix some integer valued function h with $h(n) \le n$ and write

$$g(n) = \sum_{i=1}^{h(n)} \frac{r^{n-i}}{i} + \sum_{i=h(n)+1}^{n} \frac{r^{n-i}}{i} \le h(n) r^{n-h(n)} + \frac{1}{h(n)} \frac{1 - r^{n-h(n)+2}}{1-r}\,. \tag{5.73}$$

The claim follows when taking $h(n) = \lceil n/2 \rceil$. □

Now we turn to the proof of Proposition 5.15.

Proof of Proposition 5.15. In order to work with $n \times n$ matrices, we consider the hitting time at level $n+1$. Also, we assume for simplicity that $c\rho = 1$.

By virtue of Corollary 5.13 and Lemma 5.16, we know that the vector of mean hitting times is given by $e = A^{-1} b$, where A is as in Lemma 5.16 and b contains the first moments of the waiting times, i. e.

$$b = \begin{pmatrix} 1 \\ 2^{-1} \\ \vdots \\ (n-1)^{-1} \\ n^{-1} \end{pmatrix}. \tag{5.74}$$

Each component is then given by

$$e(i) = \sum_{j=1}^{n} R_{ij}^{(n)} s_j b_j = \sum_{j=1}^{n} \frac{s_j}{j} \sum_{k=(i-j)\vee 0}^{n-j} r^k\,, \tag{5.75}$$

where $s_1 = 1$ and $s_j = q^{-1}$ for $j \ge 2$. Hence,

$$\begin{aligned} e(i) &= \sum_{j=1}^{n} \frac{s_j}{j} \sum_{k=(i-j)\vee 0}^{n-j} r^k \\ &= s_1 \sum_{k=i-1}^{n-1} r^k + \sum_{j=2}^{i} \frac{s_j}{j} \sum_{k=(i-j)}^{n-j} r^k + \sum_{j=i+1}^{n} \frac{s_j}{j} \sum_{k=0}^{n-j} r^k \\ &= \frac{1}{1-r} \left[(r^{i-1} - r^n) + s_2 \sum_{j=2}^{i} \frac{1}{j} (r^{i-j} - r^{n-j+1}) + s_2 \sum_{j=i+1}^{n} \frac{1}{j} (1 - r^{n-j+1}) \right] \\ &= \frac{1}{1-r} \left[(r^{i-1} - r^n) + s_2 \sum_{j=i+1}^{n} \frac{1}{j} + s_2 r^i \sum_{j=2}^{i} \frac{1}{j} r^{-j} - s_2 r^{n+1} \sum_{j=2}^{n} \frac{1}{j} r^{-j} \right] \end{aligned} \tag{5.76}$$

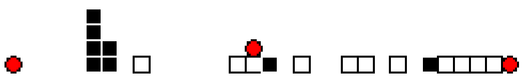
and therefore, using Lemma 5.18,

$$e(i) = \frac{1}{1-2p} \log\left(\frac{n}{i}\right) + o(1)\,, \tag{5.77}$$

if $i \to \infty$ and $i < n$.

Next, we calculate for neighboured indices $i,\ j$ the difference of the expectations:

$$\begin{aligned} e(i) - e(i+1) &= \sum_{j=1}^{n} \frac{s_j}{j} \left[\sum_{k=(i-j)\vee 0}^{n-j} r^k - \sum_{k=(i+1-j)\vee 0}^{n-j} r^k \right] \\ &= r^{i-1} + s_2 \sum_{j=1}^{i} \frac{1}{j} r^{i-j}\,. \end{aligned} \tag{5.78}$$



Recall that the first and second moment of the waiting time in state i is given by i^{-1} and $2i^{-2}$. Let g be as in Lemma 5.18. Then,

$$\begin{aligned} f(i) &= \sum_{k\in\{i-1,\, i+1\}} \frac{q(i,\,k)}{|q(i,\,i)|}\left[(e(k)-e(i))^2 + 2\frac{1}{i}\left[e(k)-e(i)\right] + \frac{2}{i^2}\right] \\ &= p\left[(r^{i-2}+g(i-1))^2 + 2\frac{1}{i}\left[r^{i-2}+g(i-1)\right] + \frac{2}{i^2}\right] \\ &\quad + q\left[(r^{i-1}+g(i))^2 + 2\frac{1}{i}\left[r^{i-1}+g(i)\right] + \frac{2}{i^2}\right]. \end{aligned} \tag{5.79}$$

Finally, to get an expression for $v(i)$ for large i, we have to calculate $A^{-1}f$. We estimate

$$v(i) = \sum_{j=1}^{n} s_j f(j) \sum_{k=(i-j)\vee 0}^{n-j} r^k \le \left[\sup_{j\le n} f(j)j^2\right]\left[\sum_{j=1}^{n}\frac{s_j}{j^2}\sum_{k=(i-j\vee 0}^{n-j} r^k\right]; \tag{5.80}$$

and we are done when we can show that the first bracket is of order $O(1)$ while the second is $o(1)$ for $i,\, n\to\infty$. Using again Lemma 5.18, it follows that

$$\begin{aligned} i^2 g(i)^2 &= O(1)\,, \\ i^2\frac{g(i)}{i} &= O(1) \qquad (i\to\infty)\,. \end{aligned} \tag{5.81}$$

Together with the geometric decay of $i\mapsto r^i$, this yields

$$\sup_{j\le n} f(j)j^2 = O(1) \qquad (n\to\infty)\,. \tag{5.82}$$

Finally (recall the calculation for the mean $e(i)$, equation (5.76)),

$$\begin{aligned} \sum_{j=1}^{n}\frac{s_j}{j^2}\sum_{k=(i-j)\vee 0}^{n-j} r^k &= \frac{1}{1-r}\left[(r^{i-1}-r^n) + s_2\sum_{j=i+1}^{n}\frac{1}{j^2} + s_2 r^i\sum_{j=2}^{i}\frac{r^{-j}}{j^2} - s_2 r^{n+1}\sum_{j=2}^{n}\frac{r^{-j}}{j^2}\right] \\ &= \frac{s_2}{1-r}\left[\sum_{j=i+1}^{n}\frac{1}{j^2} + o(i^{-1}) + O(n^{-1})\right] \\ &= \frac{s_2}{1-r}\left[\frac{1}{i} - \frac{1}{n} + O(i^{-1})\right], \qquad (i\to\infty)\,. \end{aligned} \tag{5.83}$$

□

5.5 Proof of the second assertion: Reduction of the original system

The proof of the second assertion consists of four steps:

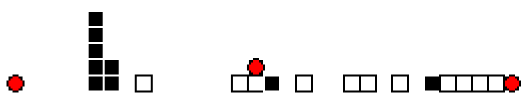
1. Rephrase the recurrence equations as given in Proposition 5.12 in terms of the N Colony System.

2. Generalize Proposition 5.12 in order to deal with more generally distributed holding times that may depend not only on the current but also on the next location.

3. Simplify the systems of equations by ignoring jumps that do not change the macrostate $[i]$ and show that the generalization of Proposition 5.12 is still applicable.

4. Simplify the system even further in order to reduce it to the harmonic random walk on $\mathbb{N}$. This is done by homogenisation of the waiting times and the step probabilities.

The details are carried out in Appendix B.3.



Part III

The second time window

The goal of this last part is to identify the evolution rules of the N Colony System in the second time window. This time window begins roughly at time $T^N_{\epsilon N}$ for some small $\epsilon > 0$. The reason is that from $T^N_{\epsilon N}$ onwards the proportion of colonies is greater or equal than ϵ, which implies that collisions become noticeable and the approximation using the Collision Free System is no longer appropriate. In order to use deterministic times, we consider from now on the time

$$t(N) = \frac{1}{\alpha} \log N + t_0 \tag{5.84}$$

for some $t_0 \in \mathbb{R}$, which is by Corollaries 2.11 and 4.14 of the same order of magnitude as $T^N_{\epsilon N}$.

Two different limits are considered:

1. We first consider the limit $t \to \infty$. Chapter 6 gives an expression for the equilibrium of the N Colony System, which is a product measure of Poisson distributions of parameter $(2s)d^{-1}$.

2. Secondly, we consider the limit $N \to \infty$. Chapter 7 identifies via a generator calculation the limiting evolution equations for

$$\left(\frac{1}{N} \Psi^N (t(N) + t) \right)_{t \geq 0} \tag{5.85}$$

in the limit $N \to \infty$. Weak convergence on path space is proved. This limiting evolution is nonlinear (due to the presence of collisions) and deterministic (due to a law of large numbers on simultaneously acting colonies) up to a random time shift that reflects the randomness in the quantity

$$T^N_{\epsilon N} - t(N)\,. \tag{5.86}$$

6 The equilibrium of the N Colony System

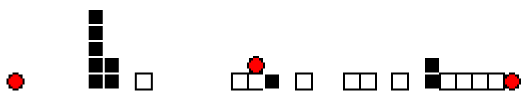
This chapter proves convergence of the time marginal distributions of the N Colony System towards a product measure of Poisson distributions in the limit $t \to \infty$. This is done in Section 6.1. Section 6.2 introduces a simplified model that can be used as an upper bound for the total number of particles Π^N. A system of a similar type will be used in Chapter 7 for certain tightness calculations. Finally, in Section 6.3, an alternative point of view is taken: It is shown that the consideration of a single colony with Poisson immigration of fixed rate leads to the same expressions for the equilibrium distribution, when the immigration rate satisfies a certain consistency condition.

6.1 The equilibrium

We state the equilibrium distribution of the N Colony System for fixed N. This distribution factorizes into its marginal distributions as if the colonies evolved independently; and the marginals neither depend on N nor on the constant of migration c.

Proposition 6.1. *For $N \in \mathbb{N}$, the equilibrium distribution π^N of the N Colony System on the state space*

$$\mathbb{S}^N = \mathbb{N}_0^N \tag{6.1}$$



is given by

$$\pi^N(k_1, ..., k_N) = \prod_{m=1}^{N} \pi^1(k_m)\,, \qquad (k_1, ..., k_N) \in \mathbb{S}^N\,, \tag{6.2}$$

where $\pi^1(k)$ is Poiss($d^{-1}2s$) distributed:

$$\pi^1(k) = \exp\left(-\frac{2s}{d}\right)\left(\frac{2s}{d}\right)^k \frac{1}{k!}\,,\ k \in \mathbb{N}_0\,. \tag{6.3}$$

Proof. It is sufficient (cf. Chapter 1 of [FK1979]) to check the detailed balance equation

$$\pi^N(k_1, ..., k_N) q\left[(k_1, ..., k_N)(l_1, ..., l_N)\right] = q\left[(l_1, ..., l_N)(k_1, ..., k_N)\right]\pi^N(l_1, ..., l_N) \tag{6.4}$$

for all N-tuples $k,\ l \in \mathbb{S}^N$, where $q(\cdot,\ \cdot)$ denotes the rate kernel. Considering an arbitrary index i, e. g. $i = 1$, this leads to

$$\binom{k_1+1}{2} d\pi^N(k_1+1,\ k_2,\ ...) = \pi^N(k_1,\ k_2,\ ...)k_1 s\ ,\ k_1 \geq 1\,, \tag{6.5}$$

and for any migration partner $j \in \{2,\ ...,\ N\}$, e. g. $j = 2$,

$$(k_1+1)\frac{c}{N}\pi^N(k_1+1,\ k_2,\ ...) = (k_2+1)\frac{c}{N}\pi^N(k_1,\ k_2+1,\ ...)\ ,\ k_1,\ k_2 \geq 0\,. \tag{6.6}$$

Both equations are satisfied by (6.2). By symmetry of the equations of the solution, all other choices of i and j lead to the same conclusion. □

Remark 6.2. *Kelly obtains similar factorising expressions for the equilibria of open and closed migration networks, cf. Chapter 2 of [FK1979]. The general conclusion is that in equilibrium quantities unexpectedly become independent. For instance, Theorem 2.2 therein states that in equilibrium the waiting times at a finite number of successive queues (of a certain type) are independent.*

By coupling, we can now obtain bounds on the number of particles Π^N. These bounds are best possible for $t \to \infty$.

Corollary 6.3. *For Π^N and any $t \in \mathbb{R}_+$, the following bound holds:*

$$\mathcal{L}\left[\Pi^N(t)\right] \leq_{st} \pi^N\,. \tag{6.7}$$

Hence,

$$\mathbb{E}\left[\Pi^N(t)\right] \leq \frac{2Ns}{d} \tag{6.8}$$

and

$$\mathbb{E}\left[(\Pi^N(t))^2\right] \leq N(N-1)\frac{2s}{d} + N\left[\frac{2s}{d} + (\frac{2s}{d})^2\right]\,. \tag{6.9}$$

Proof. The stochastic ordering is shown as in the proof of Proposition 2.8 by coupling. The coupling follows the same lines; the only obstacle is that in the realisation of π^N there must be on each colony initially more particles than there are in $\zeta^N(0)$. This is true because

$$\Pi^N(0) = 1\ ,\ \Pi^N(t) \geq 1 \text{ for all } t \geq 0\,, \tag{6.10}$$

and the initial domination is obtained by a relabelling of colonies. □

The global fluctuations of Π^N are thus under control, but this does not yet rule out the possibility of strong local fluctuations on one single colony. This is again shown by coupling.

Corollary 6.4. *Let $\zeta_i^N(t)$ denote the number of inhabitants of the i^{th} colony in the N Colony System at time t, $1 \leq i \leq N$. Then,*

$$\sup_{N\in\mathbb{N}}\ \sup_{i\leq N}\ \sup_{t\geq 0} \mathbb{E}\left[\zeta_i^N(t)^2\right] < \infty\,. \tag{6.11}$$

Proof. Repeat the coupling as sketched in the proof of Corollary 6.3, and compare the distribution with the marginal π^1. □

6.2 A simplified model without geographic structure

In Chapter 7 below, we will need to strengthen Corollary 6.4 in order to obtain bounds on the expression

$$\mathbb{E}\left[\sup_{t\leq T}\left(\zeta_1^N(t)\right)^2\right] \tag{6.12}$$

for arbitrary fixed $T > 0$. This will be part of a tightness calculation. In order to prepare for this result, we will give a simplified birth and death model $P^N(t)$ that can be used as an upper bound on the number $\Pi^N(t)$ of particles in the system. This in turn can be used to bound the number of immigrants to the fixed colony ζ_1^N. This colony can then itself be simplified to a birth and death process with a Poisson immigration stream of deterministic rate.

Proposition 6.5. *For fixed $N \in \mathbb{N}$ and constants $L \in \mathbb{N},\ D \in \mathbb{R}_+$, consider the birth and death process*

$$P^N(t)\,, \tag{6.13}$$

starting in $P^N(0) = LN$, with birth and death rates

$$P^N(t)s \text{ and } D \cdot P^N(t)1_{\{P^N(t)\geq LN+1\}} \text{ respectively.} \tag{6.14}$$

Then, the constants $L,\ D$ can be chosen such that the following hold:

1. *For any configuration of the N Colony System, the total death rate is bounded from below by the rate in the nonspatial birth and death P^N, i. e.*

$$D \cdot \Pi^N(t)1_{\{\Pi^N(t)\geq LN+1\}} \leq d\sum_{i=1}^{N}\binom{\zeta_i^N(t)}{2}\,. \tag{6.15}$$

2. *For all $t \geq 0$,*

$$\Pi^N(t) \leq_{st} P^N(t)\,. \tag{6.16}$$

3. *The process $P^N(t)$ converges for $t \to \infty$ to a unique equilibrium P^N_∞ such that*

$$P^N(t) \leq_{st} P^N_\infty\,. \tag{6.17}$$

4. *The first and second moments of the equilibrium distribution are of order $O(N)$, $O(N^2)$ respectively, for $N \to \infty$.*

Remark 6.6. *We cannot hope for a better behaviour than what is claimed under 4.: In a typical state, we expect the particles to be distributed fairly equal over the colonies; in this situation, we can neglect the geographic structure again. Then, $\Pi^N(t)$ is exposed to a death rate of about*

$$dN\binom{\lfloor \Pi^N(t)/N\rfloor}{2}\,. \tag{6.18}$$

By explicit calculation, one obtains the same asymptotical moments for this system. Moreover, when making the lower bound (6.18) *rigorous, one had to introduce a cutoff as well. We thus stick to the simpler linear cutoff defined in* (6.14) *for the ease of calculation.*

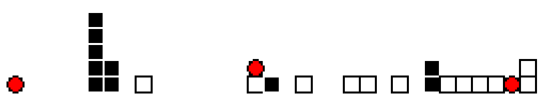
Proof of the first two assertions. From the first claim,

$$\Pi^N(t) \leq_{\text{st}} P^N(t) \tag{6.19}$$

follows immediately, because both processes have the same birth rates while the process on the right hand side starts at a higher initial value.

The claim (6.15) can be verified as follows: When $L \gg 1$ and

$$\Pi^N(t) \geq LN + 1\,, \tag{6.20}$$



then the right hand side of (6.15) is bounded from below by the death rate that is obtained from the configuration where the particles are distributed as equal as possible amongst the N colonies. This lowest possible death rate can itself be bounded from below by

$$Nd\binom{\left\lfloor \frac{\Pi^N(t)}{N} \right\rfloor}{2} \geq \frac{d}{2}(\Pi^N(t)-1)(L-1)\,, \tag{6.21}$$

and we can choose L large enough such that (6.15) holds. □

In the remainder of the proof, the process $P^N(t)$ can be considered as a simple birth and death process for which explicit equilibrium expressions are known.

Proof of the remaining assertions. Choose $D = 2s$. Because the death rate exceeds the birth rate, there exists a unique equilibrium P^N_∞, and coupling yields just as in the proof of Proposition 2.8

$$P^N(t) \leq_{\text{st}} P^N_\infty\,. \tag{6.22}$$

It remains to show that, for $N \to \infty$,

$$\begin{aligned} \mathbb{E}\left[P^N_\infty\right] &= O(N)\,, \\ \mathbb{E}\left[\left(P^N_\infty\right)^2\right] &= O(N^2)\,. \end{aligned} \tag{6.23}$$

For this, we calculate $\pi = \mathcal{L}(P^N_\infty)$ explicitly. For simplicity, we decrease the death rates by one, such that they are proportional to $(P^N - 1)$ when $P^N \geq NL + 1$. Using the usual expression for the invariant distribution of a birth and death process, one obtains for $m \geq LN$

$$\frac{\pi(m)}{\pi(LN)} = \prod_{r=\lfloor LN \rfloor+1}^{m} \frac{s(r-1)}{D(r-1)} = 2^{-m-\lfloor LN \rfloor}\,; \tag{6.24}$$

and thus, after normalization,

$$\pi(m) = 2^{-m-\lfloor LN \rfloor - 1}\,. \tag{6.25}$$

This leads to

$$\begin{aligned} \mathbb{E}\left[P^N_\infty\right] &= \sum_{m \geq \lfloor LN \rfloor} m\pi(m) \\ &= \frac{1}{2}\sum_{m\geq 0}(m+LN)\left(\frac{1}{2}\right)^m \\ &= LN + 1\,, \end{aligned} \tag{6.26}$$

and a similar expression for the second moment. □

6.3 Extension: A single colony in equilibrium with immigration

We will show that an alternative way to obtain the equilibrium distribution of the N Colony System is to consider a single colony $z(\iota, t)$ with a Poisson immigration stream of fixed rate ι. Using the consistency equation

$$\iota^* = c\mathbb{E}\left[z(\iota^*, \infty)\right]\,, \tag{6.27}$$

which states that immigration and expected emigration should be in balance in equilibrium, we regain the expression for the equilibrium as obtained in Proposition 6.1. Although this is not a formal proof, it shows that this point of view is at least consistent.

The following Proposition is an extension of Proposition 2.6 which examined a single colony in the Collision Free System. We now impose an artificial immigration stream of constant rate ι and allow the last immigrant to leave the colony. It will be argued below in Remark 6.8 that this resembles the situation in the N Colony System.

Proposition 6.7. *Let $s,\ d,\ c > 0$ and $\iota > 0$ be constants. Consider the $\mathbb{N}_0$ valued birth and death process $(z(\iota,\, t))_{t\geq 0}$ with*

$$\text{birth rate } ns + \iota \text{ and death rate } cn + \tfrac{d}{2}n(n-1)\,, \tag{6.28}$$

when there are n particles present.

- *The process $z(\iota,\, \cdot)$ converges for $t \to \infty$ towards an equilibrium distribution with equilibrium density function $(\pi(\iota,\, j) \,:\, j \in \mathbb{N}_0)$ given by*

$$\pi(\iota,\, j) = \pi(\iota,\, 0) \prod_{k=1}^{j} \frac{s(k-1)+\iota}{ck + \frac{d}{2}k(k-1)}\,. \tag{6.29}$$

  Here, $\pi(\iota,\, 0)$ is chosen such that the vector $(\pi(\iota,\, j))$ sums to 1. Furthermore, for all $k \in \mathbb{N}$,

$$\sum_{j\geq 0} j^k \pi(\iota,\, j) < \infty\,. \tag{6.30}$$

- *On $\iota \in (0,\ \infty)$, the mean of $\pi(\iota,\, \cdot)$ is increasing and continuously differentiable with respect to ι. More precisely, if a random variable $z(\iota)$ has distribution $\pi(\iota,\, \cdot)$, then*

$$\frac{d}{d\iota}\mathbb{E}\left[z(\iota)\right] \quad = \quad \mathbb{C}ov[z(\iota),\ h(z(\iota),\, \iota)]\,, \tag{6.31}$$

  where

$$h(k,\, \iota) = \sum_{m=0}^{k-1} \frac{1}{sm+\iota}\,. \tag{6.32}$$

  (The empty sum is defined to be zero.) These formulas imply that

$$\frac{d}{d\iota}\mathbb{E}[z(\iota)] \geq 0\,. \tag{6.33}$$

Proof. The claims (6.29) and (6.30) are shown as in the proof of Proposition 2.6. For differentiability, define $f(0,\, \iota) = 1$ and for $k \geq 1$

$$f(k,\, \iota) = \prod_{m=1}^{k} \frac{s(m-1)+\iota}{cm + \frac{d}{2}m(m-1)}\,. \tag{6.34}$$

Denote the m^{th} factor of this product with

$$a_m(\iota)\,. \tag{6.35}$$

Using this notation, we have

$$\pi^{(\iota)}(k) = \frac{f(k,\, \iota)}{\sum_{m\geq 0} f(m,\, \iota)}\,; \tag{6.36}$$

and, for any k, the expressions

$$f(k,\, \cdot),\ \sum_{m\geq 0} f(m,\, \cdot) \tag{6.37}$$

are differentiable: The first is a polynomial, the second a convergent power series.

Using h as defined in (6.32), we obtain

$$\frac{d}{d\iota}f(k,\, \iota) = f(k,\, \iota) \sum_{m=1}^{k} \frac{\frac{d}{d\iota}a_m(\iota)}{a_m(\iota)} = f(k,\, \iota)h(k,\, \iota)\,, \tag{6.38}$$

and thus

$$\begin{aligned}
\frac{d}{d\iota}\mathbb{E}\left[z(\iota)\right] &= \frac{d}{d\iota}\sum_{k\geq 0} k \frac{f(k,\,\iota)}{\sum_{m\geq 0} f(m,\,\iota)} \\
&= \left(\sum_{m\geq 0} f(m,\,\iota)\right)^{-2} \left\{\sum_{m\geq 0} f(m,\,\iota) \sum_{k\geq 0} k\frac{d}{d\iota} f(k,\,\iota) \right. \\
&\qquad \left. - \sum_{m\geq 0} m f(m,\,\iota) \sum_{k\geq 0} \frac{d}{d\iota} f(k,\,\iota) \right\} \\
&= \sum_{k\geq 0} k\pi^{(\iota)}(k) h(k,\,\iota) - \sum_{m\geq 0} m\pi^{(\iota)}(m) \sum_{k\geq 0} \pi^{(\iota)}(k) h(k,\,\iota)\,. \qquad (6.39)
\end{aligned}$$

This gives the asserted formula.

To see that (6.39) is nonnegative, note the integral bound

$$\frac{1}{s}\log\left(\frac{k+\frac{\iota}{s}}{1+\frac{\iota}{s}}\right) + \frac{1}{\iota} < \sum_{m=0}^{k-1} \frac{1}{sm+\iota} \qquad (6.40)$$

and apply Jensen's inequality to the convex function

$$x \mapsto \frac{1}{s}\,(x-\alpha)\log(\beta x+\gamma) + \frac{1}{\iota} \qquad (6.41)$$

where $\alpha = \mathbb{E}\left[z(\iota)\right]$, $\beta = (1+s^{-1}\iota)^{-1}$, $\gamma = s^{-1}\iota(1+s^{-1}\iota)^{-1}$. This gives

$$\begin{aligned}
\frac{d}{d\iota}\mathbb{E}\left[z(\iota)\right] &\geq \mathbb{E}\Bigg[\left(\zeta^{(\iota)}(\infty) - \mathbb{E}[\zeta^{(\iota)}(\infty)]\right) \\
&\qquad \cdot \left(\frac{1}{s}\log\left(\frac{\zeta^{(\iota)}(\infty)+s^{-1}\iota}{1+s^{-1}\iota}\right) + \frac{1}{\iota}\right)\Bigg] \\
&\geq \mathbb{E}\Bigg[\left(\zeta^{(\iota)}(\infty) - \mathbb{E}[\zeta^{(\iota)}(\infty)]\right)\Bigg] \\
&\qquad \cdot \left(\frac{1}{s}\log\left(\frac{\mathbb{E}\left[\zeta^{(\iota)}(\infty)\right]+s^{-1}\iota}{1+s^{-1}\iota}\right) + \frac{1}{\iota}\right) \\
&= 0\,. \qquad (6.42)
\end{aligned}$$

□

Remark 6.8. *As mentioned in the beginning of this section, we expect immigration and emigration to be in balance in equilibrium; the correct value of* ι *should thus satisfy the fix point equation*

$$\iota^* = c\mathbb{E}[z(\iota^*)]\,. \qquad (6.43)$$

The argument that such a ι^* *exists is the following: According to Proposition 6.7, the mapping*

$$\iota \mapsto \mathbb{E}[z(\iota)] \qquad (6.44)$$

is continuous and due to the quadratic death rate ultimately sublinear, and finally the value for $\iota = 0$ *is positive.*

This is intuitively clear but can neither be seen immediately by coupling nor analytically; already to calculate the second derivative of the mapping (6.44) *is tedious. Instead, we make the guess*

$$\iota^* = \frac{2sc}{d}\,. \qquad (6.45)$$

This choice simplifies the equations, and the corresponding normalized measure is given via

$$\pi(\iota^*,\, m) = \exp\left(-\frac{2s}{d}\right)\frac{1}{m!}\left(\frac{2s}{d}\right)^m \;; \tag{6.46}$$

it has thus a Poisson distribution with mean $d^{-1}2s$. Consequently,

$$\iota^* = \frac{2sc}{d} = c\mathbb{E}\left[z(\iota^*)\right] \;. \tag{6.47}$$

This distribution is indeed equal to the marginal distribution of the true equilibrium as calculated in Proposition 6.1.

Finally, the fact that the population size is stochastically dominated by its equilibrium remains true under immigration. We include the following extension of Proposition 2.8 as a side note and will not make use of it thereafter.

Proposition 6.9.

1. *For $(z(\iota,\, t))_{t\geq 0}$ and $z(\iota)$ as in Proposition 6.7 and any $u,\, v,\, \iota \geq 0$, the following stochastic orderings hold:*

$$z(\iota,\, 0) \leq_{st} z(\iota,\, u) \leq_{st} z(\iota,\, v) \leq_{st} z(\iota) \tag{6.48}$$

2. *The same claim holds when*

$$\iota \equiv \iota(t) \tag{6.49}$$

is a monotonically increasing function that converges to some finite value $\iota(\infty)$ for $t \to \infty$.

Proof. We consider the third stochastic ordering first. Repeat the coupling as in the proof of Proposition 2.8: Consider a population

$$(z_\infty(\iota,\, t))_{t\geq 0} \tag{6.50}$$

in equilibrium and mark one particle green. Use the same rules as before to couple its offspring, and colour additionally any immigrant green. The green population then gives a version of

$$(z(\iota,\, t))_{t\geq 0} \;, \tag{6.51}$$

while the evolution of the whole population, ignoring colours, equals

$$(z_\infty(\iota,\, t))_{t\geq 0} \;. \tag{6.52}$$

This yields the claim for fixed $\iota > 0$. In the case of a time-dependent but still deterministic immigration rate, note that by definition immigrants appear at system $z_\infty(\iota,\, \cdot)$ at rate $\iota(\infty)$. Hence, take a Poisson process of rate $\iota(\infty)$ and toss at each immigration event a coin having success probability

$$\frac{\iota(t)}{\iota(\infty)} \;; \tag{6.53}$$

if the outcome is a success, then colour the particle green; otherwise colour it white. This procedure describes independent thinning of a Poisson process of rate $\iota(\infty)$ with thinning probability

$$1 - \frac{\iota(t)}{\iota(\infty)} \;; \tag{6.54}$$

and the resulting process of green immigrants is thus an inhomogeneous Poisson process of rate $\iota(t)$, just as it should be.

The second stochastic ordering follows just as in Proposition 2.8 by comparison of the offspring of a particle that has been selected at time $v - u$ with the total population at time v. This makes it necessary to adjust the thinning of the immigrants accordingly, which is possible by monotonicity of $\iota(t)$. □

7 The nonlinear deterministic evolution

Let as before Ψ^N be the empirical statistic of the N Colony System; $\Psi^N(t,\, j)$ counts the number of colonies carrying j particles at time t. Theorem 3.6 stated that, in the initial time window starting at time $t = 0$, we have

$$\Psi^N \Rightarrow \Psi \text{ on } D\left([0,\, \infty),\, \mathcal{M}_{\text{fin}}(\mathbb{N})\right)\,, \tag{7.1}$$

where Ψ denotes the empirical statistic process of the Collision Free System. In this section, we obtain a convergence result for the second time window. The goal is to prove the following theorem. (The statement is identical to the one seen in the introduction.)

Theorem 7.1 (Convergence in the second time window)**.**
For any $t_0 \in \mathbb{R}$, there exists an $\mathcal{M}_{\leq 1}(\mathbb{N})$ valued continuous process $(\Phi(t))_{t\in\mathbb{R}}$ such that

$$\left(\frac{1}{N}\Psi^N((\frac{1}{\alpha}\log N + t_0 + t) \vee 0)\right)_{t\in\mathbb{R}} \Rightarrow (\Phi(t))_{t\in\mathbb{R}} \tag{7.2}$$

on the path space $D(\mathbb{R},\, \mathcal{M}_{\leq 1}(\mathbb{N}))$. The process Φ has the following properties:

1. *The process is deterministic up to a random time shift, i. e.*

$$(\Phi(\tau(\epsilon) + t))_{t\in\mathbb{R}} \tag{7.3}$$

   is a deterministic process, when $\tau(\epsilon)$ denotes the first passage time of $\Phi(t,\, \mathbb{N})$ at level $\epsilon \in (0,\, 1)$.

2. *This random time shift is only caused by the randomness collected in the first collision free time window. More precisely, there exists a deterministic function $\tilde{\tau}(\epsilon)$ such that*

$$\tau(\epsilon) \overset{d}{=} \tilde{\tau}(\epsilon) + \frac{1}{\alpha}\log\frac{\epsilon}{W}\,, \tag{7.4}$$

   where the summand $\alpha^{-1}\log\epsilon W^{-1}$ can be identified with the limit of the normalized hitting times $T_{\epsilon N} - \alpha^{-1}\log N$ of the Collision Free System.

3. *Given any initial point Φ_0, the proportion of occupied colonies $\Phi(t,\, \mathbb{N})$ satisfies a logistic differential equation with time dependent coefficients:*

$$\frac{d}{dt}\Phi(t,\, \mathbb{N}) = \alpha(\Phi(t,\, \cdot))\Phi(t,\, \mathbb{N})\left(1 - \left(1 + \frac{\gamma(\Phi(t,\, \cdot))}{\alpha(\Phi(t,\, \cdot))}\right)\Phi(t,\, \mathbb{N})\right)\,, \tag{7.5}$$

   where

$$\gamma(\Phi(t,\, \cdot)) = c\Phi(t,\, 1), \qquad \alpha(\Phi(t,\, \cdot)) = c\sum_{k\geq 2} k\Phi(t,\, k)\,. \tag{7.6}$$

   Also, the components $\{\Phi(t,\, j)\,,\, j \in \mathbb{N}\}$ satisfy a coupled system of differential equations that can be specified explicitly.

4. *If $t_0 = 0$, then*

$$\lim_{t\to-\infty}\frac{\Phi(t,\, \mathbb{N})}{\exp(\alpha t)} \overset{d}{=} W\,, \tag{7.7}$$

   where W is the growth variable that was specified in Theorem 1.11.

In Section 7.1, the preparations for the following generator calculations are carried out. The actual proof is presented in Section 7.2.

7.1 Preparations

This section aims to set the stage for the oncoming analysis by introducing suitable spaces and test functions. We first introduce abbreviations for the process and the time shift.

Definition 7.2 (Time shift and normalizations)**.**
Define for given $t_0 \in \mathbb{R}$ *the* time shift

$$t(N) = \frac{1}{\alpha} \log N + t_0 \tag{7.8}$$

and define the normalized statistics *via*

$$\hat{\Psi}^N(t, \cdot) = \frac{\Psi^N(t, \cdot)}{N} . \tag{7.9}$$

In Subsection 7.1.1 below, the underlying space of $\hat{\Psi}^N$ is examined. In Subsection 7.1.2, the test functions living on this space are introduced. Finally, in Subsection 7.1.3, the action of the generator of $\hat{\Psi}^N$ on these test functions is stated.

7.1.1 The state space

Recall the definition

$$\mathcal{M}_{\leq 1}(\mathbb{N}) = \{\psi \in [0, 1]^{\mathbb{N}} \ : \ \sum_{k \geq 1} \psi(k) \leq 1\} . \tag{7.10}$$

This set is endowed with the topology of weak convergence and can be identified with a subset of

$$L^1_\lambda(\mathbb{N}) , \tag{7.11}$$

where λ denotes the counting measure on $\mathbb{N}$ (cf. Appendix A.3).

Definition 7.3 (subsets of the state space)**.**
Define for given $B \in \mathbb{R}_+$ *the sets of* measures with finite second *or* fourth moment *and* bounded second moment *respectively via*

$$\begin{aligned} \mathcal{M}^2_{\leq 1}(\mathbb{N}) &= \{\psi \in \mathcal{M}_{\leq 1}(\mathbb{N}) \ : \ \sum_{k \geq 1} k^2 \psi(k) < \infty\} , \\ \mathcal{M}^4_{\leq 1}(\mathbb{N}) &= \{\psi \in \mathcal{M}_{\leq 1}(\mathbb{N}) \ : \ \sum_{k \geq 1} k^4 \psi(k) < \infty\} , \\ \mathcal{M}^{2;B}_{\leq 1}(\mathbb{N}) &= \{\psi \in \mathcal{M}_{\leq 1}(\mathbb{N}) \ : \ \sum_{k \geq 1} k^2 \psi(k) \leq B\} . \end{aligned} \tag{7.12}$$

Endow these sets with the subset topology of $\mathcal{M}_{\leq 1}(\mathbb{N})$*.*

We show that $\mathcal{M}^{2;B}_{\leq 1}(\mathbb{N})$ is compact. In Subsection 7.2.1, we will make use of this property in order to prove a compact containment statement.

Lemma 7.4. *For any* $B \in \mathbb{R}_+$*, the set* $\mathcal{M}^{2;B}_{\leq 1}(\mathbb{N})$ *is compact in the topology of weak convergence.*

Proof. Consider an arbitrary sequence $\{\mu_i\} \subset \mathcal{M}^{2;B}_{\leq 1}(\mathbb{N})$. We show that it is tight: Let some $\epsilon > 0$ be given. Define

$$M = \left\lceil \sqrt{\frac{B}{\epsilon}} \right\rceil . \tag{7.13}$$

Then,

$$\mu_i(\{M,\, M+1,\, ...\}) \le \frac{1}{M^2}\sum_{k\ge M} k^2\mu_i(k) \le \frac{B}{M^2} \le \epsilon\,, \tag{7.14}$$

uniformly in i. Hence, the sequence is tight and, since $\mathbb{N}$ is Polish, relatively compact. Now let μ be an accumulation point of the sequence and assume for simplicity that

$$\mu_i \Rightarrow \mu\,. \tag{7.15}$$

If $\mu \notin \mathcal{M}^{2,B}_{\le 1}(\mathbb{N})$, this would imply that there exists an $M \in \mathbb{N}$ such that

$$\sum_{k=1}^{M} k^2\mu(k) > B\,. \tag{7.16}$$

But the left hand side is the limit of the corresponding quantities for the measures μ_i which are bounded by B. This is a contradiction. □

Remark 7.5. *It will be necessary below for the generator calculations to restrict the state space to $\mathcal{M}^2_{\le 1}(\mathbb{N})$. The reason why we do not choose $\mathcal{M}^2_{\le 1}(\mathbb{N})$ as state space in the first place is because it is not complete: For instance, the sequence*

$$\psi_n = \frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{k^2}\delta_k \in \mathcal{M}^2_{\le 1}(\mathbb{N}) \tag{7.17}$$

converges weakly towards the measure

$$\psi = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\delta_k \in \mathcal{M}_1(\mathbb{N}) \setminus \mathcal{M}^2_{\le 1}(\mathbb{N})\,. \tag{7.18}$$

7.1.2 The test functions

Finally, we introduce test functions acting on the set $\mathcal{M}_{\le 1}(\mathbb{N})$.

Definition 7.6 (The test functions)**.**
Define the set of test functions

$$\mathcal{F} \tag{7.19}$$

as the set of functions $F_{f,g} : \mathcal{M}_{\le 1}(\mathbb{N}) \to \mathbb{R}$ that can be written as follows:

$$F_{f,g}(\psi) = g\left(\sum_{k\ge 1} f(k)\psi(k)\right)\,. \tag{7.20}$$

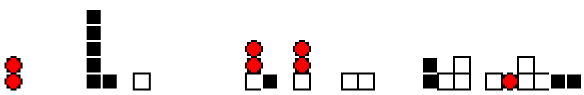
Here, $f : \mathbb{N} \to \mathbb{R}$ is bounded, and $g : \mathbb{R} \to \mathbb{R}$ is continuous and additionally twice continuously differentiable with bounded first and second derivatives. Such f and g are called admissible*.*

Remark 7.7. *It is noteworthy that, for given $F \in \mathcal{F}$, there are several choices of admissible f and g such that $F = F_{f,g}$; for instance,*

$$F_{f,g} = F_{Cf,g(C^{-1}\cdot)} \tag{7.21}$$

for any $C \neq 0$. Below, the action of the generator of $\hat{\Psi}^N$ on a given function $F \in \mathcal{F}$ is stated in terms of its ingredients f, g, and we thus have to make sure that this does not depend on the representative $F_{f,g}$.

It is necessary for the generator calculations below that the set of functions is at least separating. This is the content of the following Lemma.



Lemma 7.8. *The set $\mathcal{F}$ is separating, i. e. for measures μ_1, $\mu_2 \in \mathcal{M}_1(\mathcal{M}_{\leq 1}(\mathbb{N}))$ the property*

$$\int F(\psi)\mu_1(d\psi) = \int F(\psi)\mu_2(d\psi) \qquad \text{for all } F \in \mathcal{F} \tag{7.22}$$

implies $\mu_1 = \mu_2$.

Proof. It is sufficient that the set

$$\tilde{\mathcal{F}} = \{\langle f, \cdot\rangle \,:\, f \in C_b(\mathbb{N}, \mathbb{R})\} \tag{7.23}$$

is separating, where we abbreviate

$$\langle f, \psi\rangle = \sum_{k\geq 1} f(k)\psi(k)\,. \tag{7.24}$$

The latter follows from the fact that the algebra $\tilde{\mathcal{F}}$ separates points (cf. Theorem 3.4.5 of [EK1986]), which can be seen as follows: Suppose ϕ, $\psi \in \mathcal{M}_{\leq 1}(\mathbb{N})$ and $\phi \neq \psi$. If

$$\langle f, \psi\rangle = \langle f, \phi\rangle \quad \text{for all } f \in \tilde{\mathcal{F}}\,, \tag{7.25}$$

then it would follow that $\psi = \phi$, since $\tilde{\mathcal{F}}$ is separating for measures on $\mathbb{N}$. Hence, there must be some f such that the equality in (7.25) is violated. □

Remark 7.9. *The set $\mathcal{F}$ is itself not an algebra because the functions g need not to be multiplicative. But it harbours two subsets that are algebras:*

- *The set*
$$\mathcal{F}_1 = \{\exp(\langle f, \cdot\rangle \,:\, f \in C_b(\mathbb{N}, \mathbb{R})\} \tag{7.26}$$
is a separating algebra. Since f is bounded, any such function is still bounded by $\exp(||f||_\infty)$. The derivatives can be made bounded by suitable truncations.

- *For fixed f, the set*
$$\mathcal{F}_{2,f} = \{F_{\tilde{f},g} \in \mathcal{F} \,:\, \tilde{f} = f\} \tag{7.27}$$
is an algebra because monomials of admissible g are still admissible.

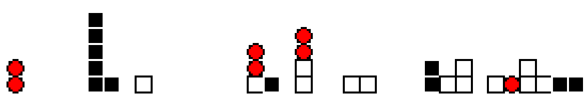
7.1.3 The generator of $\hat{\Psi}^N$

Having defined a suitable set of test functions, the next task is to identify the generator of $\hat{\Psi}^N$ when restricted to this set of functions. We use in the following the abbreviation

$$\psi(f) = \sum_{j\geq 1} \psi(j)f(j)\,. \tag{7.28}$$

We can consider $\hat{\Psi}^N$ as a process taking values in $\mathcal{M}^2_{\leq 1}(\mathbb{N})$ because of Corollary 6.4.

Lemma 7.10. *Let $\mathcal{F}$ be as above. The generator G^N of the $\mathcal{M}^2_{\leq 1}(\mathbb{N})$-valued Markov process $\hat{\Psi}^N$ acting on functions $F \in \mathcal{F}$ is given by the following expression, which is in particular*



finite and does not depend on the choice of representation $\{f, g\}$ *for* $F_{f,g}$*:*

$$
\begin{aligned}
G^N F_{f,g}(\psi) \quad = \quad & \sum_{j\geq 1} \psi(j) \\
& \cdot \Bigg[sNj \left(g\Big(\psi(f) + \frac{1}{N}(f(j+1) - f(j))\Big) - g\Big(\psi(f)\Big) \right) \\
& + \frac{d}{2} Nj(j-1) \left(g\Big(\psi(f) + \frac{1}{N}\left[f(j-1) - f(j)\right]\Big) - g\Big(\psi(f)\Big) \right) \\
& + 1_{\{j\geq 2\}} cNj(1 - \psi(\mathbb{N})) \\
& \quad \cdot \left(g\Big(\psi(f) + \frac{1}{N}(f(1) + f(j-1) - f(j))\Big) - g\Big(\psi(f)\Big) \right) \\
& + cNj \sum_{k\geq 1} \psi(k) \\
& \quad \cdot \bigg(g\Big(\psi(f) + \frac{1}{N}(f(j-1)1_{\{j>1\}} - f(j)) \\
& \qquad + \frac{1}{N}(f(k+1) - f(k))\Big) - g\Big(\psi(f)\Big) \bigg) \Bigg] . \qquad (7.29)
\end{aligned}
$$

Proof. Let $\delta_i \in \mathcal{M}_1(\mathbb{N})$ denote the Dirac measure on $i \in \mathbb{N}$. The transition rates of the jump process $\hat{\Psi}^N$ are as follows:

- Transitions that are caused by births and deaths: For $j \geq 1$, we have

$$
\begin{aligned}
\psi \mapsto \psi + \frac{1}{N}(\delta_{j+1} - \delta_j) \qquad & \text{at rate} \quad N\psi(j)sj \ , \\
\psi \mapsto \psi + \frac{1}{N}(\delta_j - \delta_{j+1}) \qquad & \text{at rate} \quad N\psi(j+1)\frac{d}{2}j(j+1) \ . \qquad (7.30)
\end{aligned}
$$

- Transitions that are caused by migrations from a colony with j particles to a colony with k particles: For $j \geq 1$ and $k \geq 1$, we have

$$
\begin{aligned}
\psi \mapsto \psi + \frac{1}{N}(\delta_{j-1}1_{\{j>1\}} - \delta_j & \\
+\delta_{k+1} - \delta_k) \qquad & \text{at rate} \quad (Nc\psi(j)j)\frac{N\psi(k)}{N} \ , \qquad (7.31)
\end{aligned}
$$

  and for $k = 0$ and $j \geq 2$,

$$
\psi \mapsto \psi + \frac{1}{N}(\delta_{j-1} - \delta_j + \delta_1) \qquad \text{at rate} \quad (Nc\psi(j)j)\frac{(N - N\psi(\mathbb{N}))}{N} \ . \qquad (7.32)
$$

  If $j = 1$, then the jump in (7.32) goes unnoticed because it does not change the state. This corresponds to a non-colliding single particle migration.

The representation (7.29) then follows from the fact that

$$
(\psi + \frac{1}{N}\delta_j)(f) = \sum_{i\geq 1} \left(\psi(i) + \frac{1}{N}\delta_j(i)\right) f(i) = \psi(f) + \frac{1}{N}f(j) \, . \qquad (7.33)
$$

Finally, being defined as a sum over $F_{f,g}(\cdot)$ evaluated at different points, it is immediately clear that (7.29) does not depend on the choice of $\{f, g\}$. □

7.2 Proof of Theorem 7.1

We apply Corollary 4.8.16 of [EK1986] which is summarized in Theorem A.11 in Appendix A.4. The following subchapters are devoted to the verification of the conditions of this convergence criterion.

We have to show the following:

1. There exists a sequence of compact sets C^N such that

$$\lim_{N\to\infty} \mathbb{P}\left(\hat{\Psi}^N(t(N)+t) \in C^N \text{ for all } t \in [0, T]\right) = 1 \tag{7.34}$$

and, almost equivalently, such that for all $\eta > 0$ there exists a $N_0 = N_0(\eta)$ satisfying

$$\inf_{N\in\mathbb{N}} \mathbb{P}\left(\hat{\Psi}^N(t(N)+t) \in C^{N_0} \text{ for all } t \in [0, T]\right) \geq 1-\eta\,. \tag{7.35}$$

This will be proved in Subsection 7.2.1 with the choice

$$C^N = \mathcal{M}^{2;\sqrt{N}}_{\leq 1}(\mathbb{N})\,. \tag{7.36}$$

2. There exists a limit generator G of the sequence $\{G^N : N \in \mathbb{N}\}$ such that the convergence is uniform on the sets C^N: for any $F \in \mathcal{F}$,

$$\lim_{N\to\infty} \sup_{\psi\in C^N} |(G^N F)(\psi) - (GF)(\psi)| = 0\,. \tag{7.37}$$

This is shown in Subsection 7.2.2.

The first assertion also implies that the sequence

$$\{\hat{\Psi}^N(t(N)) \,:\, N \in \mathbb{N}\} \tag{7.38}$$

of initial conditions is tight. In this step of the argument, an accumulation point

$$\mu \tag{7.39}$$

of this sequence is fixed. In the final fifth step below it will then be argued that, using vaguely speaking a comparison with the Collision Free System at time $t = -\infty$, this measure must be unique.

3. The martingale problem associated to (G, μ) has at most one solution if a certain system of differential equations is uniquely solvable. This implication is shown in Subsection 7.2.3.

4. The system of differential equation is indeed uniquely solvable. This is shown by inferring general Banach valued differential equation theory in Subsection 7.2.4.

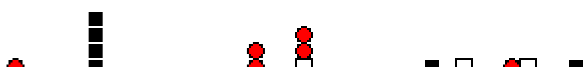
The steps 1-4 together with Theorem A.11 then imply that the processes

$$\left\{\left(\hat{\Psi}^{N_k}(t(N_k)+t)\right)_{t\geq 0} \,:\, k \in \mathbb{N}\right\} \tag{7.40}$$

converge towards a deterministic process Φ with (pre)generator G along the subsequence $k \mapsto N_k$ for which we know weak convergence of the initial condition. The proof is finished by showing that the form of Φ only allows one accumulation point of (7.38).

5. The sequence (7.38) can at most have one accumulation point. This is done in Subsection 7.2.5.

The additional properties of Φ as mentioned in Theorem 7.1 also follow from the discussion in this fifth step. This finishes the proof of Theorem 7.1.



7.2.1 Step 1: A compact containment condition for $\mathcal{L}(\hat{\Psi}^N)$

The goal of this section is that on any bounded time interval the second moment of the normalized size distribution $\hat{\Psi}^N$ does not get unbounded for $N \to \infty$.

Lemma 7.11. *For any $\eta > 0$, $T > 0$, there exists a constant $B \in \mathbb{R}_+$ such that*

$$\sup_{N\in\mathbb{N}} \mathbb{P}\left(\sup_{t\in[0,\,T]} \sum_{k\geq 1} k^2 \hat{\Psi}^N(t(N)+t,\, k) \geq B\right) < \eta\,. \tag{7.41}$$

We defer the proof of the lemma to the end of the section and consider the consequences first.

Consequences. Both compact containment conditions (7.34) and (7.35) follow from this result. Recall that the set $\mathcal{M}^{2,B}_{\leq 1}(\mathbb{N})$ is compact in the topology of weak convergence, cf. Lemma 7.4.

Corollary 7.12. *It follows that for all T*

$$\lim_{N\to\infty} \mathbb{P}\left(\hat{\Psi}^N(t(N)+t) \in \mathcal{M}^{2,\sqrt{N}}_{\leq 1}(\mathbb{N}) \text{ for all } t\in[0,\,T]\right) = 1 \tag{7.42}$$

and for all $\eta > 0$ there exists an $N_0 \in \mathbb{N}$ such that

$$\inf_{N\in\mathbb{N}} \mathbb{P}\left(\hat{\Psi}^N(t(N)+t) \in \mathcal{M}^{2,\sqrt{N_0}}_{\leq 1}(\mathbb{N}) \text{ for all } t\in[0,\,T]\right) \geq 1-\eta\,. \tag{7.43}$$

Proof. This follows from the definition of $\mathcal{M}^{2,\sqrt{N}}_{\leq 1}(\mathbb{N})$. □

Also, tightness is a consequence of Lemma 7.11. We quickly state the following corollary and defer its proof to Appendix B.4 since we do not make explicit use of the statement (of course, tightness is implicitly employed somewhere in the machinery of the convergence criterion that is taken from [EK1986]).

Corollary 7.13. *The sequence*

$$\mathcal{L}\left[\left(\hat{\Psi}^N(t(N)+t)\right)_{t\geq 0}\right] \tag{7.44}$$

of measures on $D([0,\,\infty),\, \mathcal{M}_{\leq 1}(\mathbb{N}))$ is relatively compact.

We use a standard criterion which states that it is sufficient to prove compact containment and that the modulus of continuity is small in a certain sense. Lemma 7.11 can be used to bound the transition rates; this in turn can be used to bound the modulus of continuity with the corresponding quantity of a Poisson process of rate $O(N)$ with jump sizes $O(N^{-1})$. The latter is easily shown to be small. The proof is carried out in Appendix B.4.

Proof of Lemma 7.11. We now turn to the proof. We first reduce the problem to the integrability of a certain function. It is then shown below in two subsequent Lemmas that the growth of the function can be controlled while the tail of the integrating measure μ falls off quickly enough.

Lemma 7.14. *For fixed B, the following holds:*

$$\mathbb{P}\left(\sup_{t\leq T} \sum_{k\geq 1} k^2 \hat{\Psi}^N(t(N)+t,\, k) \geq B\right) \leq \frac{1}{B}\int_{\mathbb{R}_+} \mathbb{E}\left[\sup_{t\leq T}(z_\iota(t))^2\right] \mu(d\iota)\,. \tag{7.45}$$

Here, z_ι is a birth and death process starting in its equilibrium with

$$\textit{birth rates } sn+\iota \quad \textit{and} \quad \textit{death rates} \quad 2sn1_{\{n>L(\iota)\}}\,, \tag{7.46}$$

where $L(\iota)$ is the smallest positive integer that satisfies

$$\left((k+1)^2-k^2\right)(sk+\iota)+\left((k-1)^2-k^2\right)d\binom{k}{2}<0 \quad \textit{for all } k>L(\iota)\,. \tag{7.47}$$

The measure μ is the distribution of

$$\sup_{t\le T}\frac{c}{N}Z(t)\,, \tag{7.48}$$

where Z is a birth and death process that is independent of z_ι, also starting in equilibrium. The birth and death rates are

$$sn \quad \textit{and} \quad 2sn1_{\{n>(N-1)L(0)\}} \quad \textit{respectively.} \tag{7.49}$$

The integer $L(0)$ is defined as in expression (7.47) *when $\iota=0$.*

Proof. We use the identity

$$\sum_{k\ge1}k^2\hat\Psi^N(t(N)+t,\,k)=\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_i^N(t(N)+t)\right)^2 \tag{7.50}$$

and the stochastic ordering

$$\sup_{t\le T}\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_i^N(t(N)+t)\right)^2\le_{\text{st}}\sup_{t\le T}\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_i^N(\infty,\,t)\right)^2\,, \tag{7.51}$$

where $\zeta_i^N(\infty,\cdot)$ denotes the process started in its stationary law. Also, interchanging the sup and the sum increases the expression. By exchangeability, we have

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sup_{t\le T}\left(\zeta_i^N(\infty,\,t)\right)^2\right]=\mathbb{E}\left[\sup_{t\le T}\left(\zeta_1^N(\infty,\,t)\right)^2\right]\,. \tag{7.52}$$

Hence, the task is to find bounds for the population number on one colony. We estimate as follows (the newly introduced quantities are explained below):

$$\sup_{t\le T}\left(\zeta_1^N(\infty,\,t)\right)^2\le_{\text{st}}\sup_{t\le T}\left(z\Big(\frac{c}{N}\int_0^t Z(u)du,\,t\Big)\right)^2\,. \tag{7.53}$$

Here, we modify

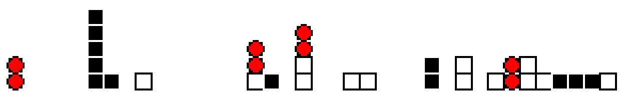
- the birth and death rules in the whole system: Replace the death rate in ζ_1^N

  $$d\binom{n}{2} \quad \text{by the rate} \quad 2sn1_{\{n>L(0)\}}\,, \tag{7.54}$$

  where the constant $L(0)$ is as in expression (7.47). Make the death rates in the surrounding $N-1$ colonies independent of the local occupancy numbers and replace it with the global rate

  $$2sn1_{\{n>(N-1)L(0)\}}\,. \tag{7.55}$$

- the migration rules: Any emigration from colony 1 is now forbidden, and any immigrant from the surrounding $N-1$ colonies now does not leave its original colony but is copied into the first. Both rules increase the population size in the first colony and have the effect that the first colony does not influence its environment. Hence, the environment is given by the process Z as described around expression (7.49).



The first parameter of z is the accumulated immigration rate up to time t which is caused by Z. The latter process is an independently evolving population on a larger geographic space.

For fixed $\iota_0 > 0$, consider the set

$$\left\{ \sup_{t \le T} \frac{c}{N} Z(t) \le \iota_0 \right\} . \tag{7.56}$$

Making use of the fact that Z evolves independently of z, we can on this set even further simplify the component z and replace it by

$$z_{\iota_0} \tag{7.57}$$

with the following modifications: the Poisson immigration stream has fixed rate ι_0, and the constant $L(0)$ that decides where the death rates begin is adjusted to some $L(\iota_0)$ as in expression (7.47).

Putting all these orderings together, the disintegration formula yields

$$\mathbb{P}\left(\sup_{t \le T} \sum_{k \ge 1} k^2 \hat{\Psi}^N(t(N) + t,\, k) \ge B \right) \le \frac{1}{B} \int_{\mathbb{R}_+} \mathbb{E}\left[\sup_{t \le T} (z_\iota(t))^2 \right] \mu(d\iota)\,, \tag{7.58}$$

where μ is the distribution of $\sup_{u \le T} \frac{c}{N} Z(u)$ on $\mathbb{R}_+$. □

We have to show that the integral in Lemma 7.14 is finite. The strategy is to show that the integrand is of order $o(\iota^3)$ while the measure μ has arbitrarily light tails. The argument is that, by construction, the processes z, Z are supermartingales when accordingly centred; the centring is necessary in order to remove the Poisson drift that points upwards when the processes are in their lowest state.

Lemma 7.15. *Let z be as in Lemma 7.14. Then,*

$$\mathbb{E}\left[\sup_{t \in [0,\, T]} (z_\iota(t))^2 \right] = o(\iota^3) \qquad (\iota \to \infty)\,. \tag{7.59}$$

Proof. An upper bound for the constant $L(\iota)$ as defined in (7.47) is obtained by assuming a priori $L(\iota) \ge \iota^{1/3}$ and solving

$$-(x-1)^2 d + 3(x-1)s + \frac{\iota}{\iota^{1/3}} = 0\,. \tag{7.60}$$

(Here, we used the estimates $(-2x+1)(x-1)/2 \le -(x-1)^2$ and $(2x+1) \le 3(x-1)$ for $x \ge 4$). This leads to

$$L(\iota) = \left\lceil \frac{1}{d}(3s + \sqrt{9s^2 + 4d\iota^{\frac{2}{3}}}) \right\rceil \vee 4 \vee \left\lceil \iota^{\frac{1}{3}} \right\rceil = O(\iota^{1/3})\,, \qquad (\iota \to \infty)\,. \tag{7.61}$$

Define

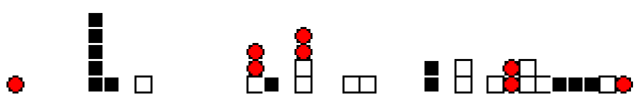
$$C(\iota) = L(\iota)s + \iota\,. \tag{7.62}$$

It follows immediately that

$$z_\iota(t)^2 - C(\iota)t \tag{7.63}$$

is a supermartingale (this can for instance be seen by considering the Markov process $(z_\iota(\cdot),\, \cdot)$ for which the function $h(x,\, t) = x^2 - C(\iota)t$ is superharmonic). Using the fact that stopped supermartingales are supermartingales, we obtain, when considering the hitting time $\tau(M)$ of the process (7.63) at level $M \in \mathbb{R}^+$,

$$\mathbb{E}\left[\sup_{t \le T \wedge \tau(M)} \left[z_\iota(t)^2 - C(\iota)t \right] \right] \le \mathbb{E}\left[z_\iota(\tau_M)^2 - C(\iota)\tau_M \right] \le \mathbb{E}\left[(z_\iota(0))^2 \right] . \tag{7.64}$$



Using monotone convergence and that $\tau(M) \to \infty$ by the nonexplosion property, we obtain the assertion for the centred process; we use here that the second moment of the equilibrium of z_ι grows at most quadratically in ι (when we neglect the dependency of $L(\iota)$ on ι). The assertion then follows from the inequality

$$\mathbb{E}\left[\sup_{t\leq T} z_\iota(t)^2\right] \leq \mathbb{E}\left[\sup_{t\leq T}\left[z_\iota(t)^2 - C(\iota)t\right]\right] + C(\iota)T\,. \tag{7.65}$$

□

We finally turn to the measure μ, i. e. the tail of the distribution of the larger population Z.

Lemma 7.16. *For any k, there exist constants A, $B > 0$ such that*

$$\mathbb{P}\left(\sup_{t\leq T}\frac{c}{N}Z(t) > \iota\right) \leq \frac{A}{\iota^k} \qquad \textit{for all } \iota > B\,. \tag{7.66}$$

Proof. This follows by repeating the previous centring argument such that $Z(t)^k - C(k)t$ becomes a supermartingale (for small N, the constant $L(0)$ must then be modified to some $L^{(k)}(0)$ such that the drift points downwards for all $m > (N-1)L^{(k)}(0)$ even with the weights $(m+1)^k - m^k$) where $C(k) = O(N^k)$. Another simpler strategy is to bound $Z(t)$ by a pure birth process of rate s which also has finite k^{th} moment of order $O(N^k)$ after time T. □

This finishes the proof of Lemma 7.11.

7.2.2 Step 2: Uniform convergence towards the limit generator

We identify the generator of the potential limit process.

Lemma 7.17. *The generator G^N as identified in Lemma 7.10 converges towards some G which is given as follows:*

$$\begin{aligned} GF_{f,g}(\psi) \;=\; & g'(\psi(f))\sum_{j\geq 1}\psi(j) \\ & \Bigg[sj\,[f(j+1)-f(j)] \\ & +1_{\{j>1\}}\frac{d}{2}j(j-1)\,[f(j-1)-f(j)] \\ & +1_{\{j>1\}}cj(1-\psi(1_{\mathbb{N}}))\,[f(1)+f(j-1)-f(j)] \\ & +cj\psi(1_{\mathbb{N}})\left[f(j-1)1_{\{j>1\}}-f(j)\right] \\ & +c\left(\sum_{k\geq 1}k\psi(k)\right)[f(j+1)-f(j)]\Bigg]\,. \end{aligned} \tag{7.67}$$

For fixed $F_{f,g} \in \mathcal{F}$, the convergence is uniform on the set $\mathcal{M}_{\leq 1}^{2,\sqrt{N}}(\mathbb{N})$, i. e.

$$\lim_{N\to\infty}\sup_{\psi\in\mathcal{M}_{\leq 1}^{2,\sqrt{N}}(\mathbb{N})}\left|(G^N F_{f,g})(\psi) - (GF_{f,g})(\psi)\right| = 0\,. \tag{7.68}$$

Again, the right hand side of (7.67) *does not depend on the choice of representation* $\{f,\, g\}$.

Proof. Fix some admissible pair $\{f, g\}$ and some $\psi \in \mathcal{M}^2_{\leq 1}(\mathbb{N})$. Expression (7.67) is equivalent to the following:

$$\begin{aligned} GF_{f,g}(\psi) &= \sum_{j\geq 1} \psi(j) g'(\psi(f)) \\ & \Bigg[sj\left([f(j+1) - f(j)]\right) \\ & + 1_{\{j>1\}} \frac{d}{2} j(j-1) \left[f(j-1) - f(j)\right] \\ & + 1_{\{j>1\}} cj(1-\psi(1_{\mathbb{N}})) \left[f(1) + f(j-1) - f(j)\right] \\ & + cj \sum_{k\geq 1} \psi(k) \left[f(j-1)1_{\{j>1\}} - f(j) + f(k+1) - f(k)\right] \Bigg]. \end{aligned} \quad (7.69)$$

This follows since separating the sum in the last line of (7.69) yields the expression

$$c\left(\sum_{k\geq 1}\psi(k)\right)\sum_{j\geq 1}\psi(j)j\left[f(j-1)1_{\{j>1\}} - f(j)\right] + c\left(\sum_{j\geq 1}\psi(j)j\right)\sum_{k\geq 1}\psi(k)\left[f(k+1) - f(k)\right]; \quad (7.70)$$

interchanging the summation indices in the second summand finally gives (7.67). We will now compare (7.69) term by term with the generator G^N. Write

$$G^N(\cdot)(\psi) = \sum_{j\geq 1}\psi(j)G^N_j(\cdot)(\psi), \qquad G(\cdot)(\psi) = \sum_{j\geq 1}\psi(j)G_j(\cdot)(\psi), \quad (7.71)$$

where the mappings

$$G^N_j,\, G_j \,:\, \mathcal{F}\times\mathcal{M}^2_{\leq 1} \to \mathbb{R} \quad (7.72)$$

are given by the j^{th} summand in the expressions (7.29) and (7.67) respectively. Then,

$$\left|G^N F_{f,g}(\psi) - GF_{f,g}(\psi)\right| \leq \sum_{j\geq 1}\psi(j)\left|G^N_j F_{f,g}(\psi) - G_j F_{f,g}(\psi)\right|, \quad (7.73)$$

and we are done when we can show that this is small uniformly in ψ. We can further divide (7.73) into the summands $G^N_{j,s}$, $G^N_{j,d}$, $G^N_{j,c}$ that correspond to birth, death and migration events. The usual Taylor approximations lead to

$$\begin{aligned} \left|G^N_{j,s}F_{f,g}(\psi) - G_{j,s}F_{f,g}(\psi)\right| &\leq \frac{\text{const}}{N} j \sup_{x\in\mathbb{R}} g''(x), \\ \left|G^N_{j,d}F_{f,g}(\psi) - G_{j,d}F_{f,g}(\psi)\right| &\leq \frac{\text{const}}{N} j^2 \sup_{x\in\mathbb{R}} g''(x), \\ \left|G^N_{j,c}F_{f,g}(\psi) - G_{j,c}F_{f,g}(\psi)\right| &\leq \frac{\text{const}}{N} j \sup_{x\in\mathbb{R}} g''(x), \end{aligned} \quad (7.74)$$

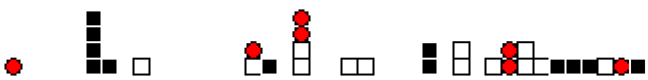
where the constants do not depend on j or ψ. This yields

$$\left|G^N F_{f,g}(\psi) - GF_{f,g}(\psi)\right| \leq \frac{\text{const}}{N}\sum_{j\geq 1}\psi(j)j^2, \quad (7.75)$$

and thus

$$\sup_{\psi\in\mathcal{M}^{2,\sqrt{N}}_{\leq 1}(\mathbb{N})} \left|G^N F_{f,g}(\psi) - GF_{f,g}(\psi)\right| \leq \frac{\text{const}\cdot\sqrt{N}}{N} \to 0. \quad (7.76)$$

□



We can immediately read off that the limiting process, if it exists, is deterministic.

Corollary 7.18. *Any collection of Markov processes* $\{\mathbb{E}_x : x \in \mathcal{M}_{\leq 1}(\mathbb{N})\}$ *on* $\mathcal{M}_{\leq 1}(\mathbb{N})$ *satisfying*

$$\lim_{t \to 0} \frac{1}{x} \left(\mathbb{E}_t \left[F_{f,g}(X(t)) \right] - F_{f,g}(x) \right) = (GF_{f,g})(x) \tag{7.77}$$

for all $x \in \mathcal{M}^2_{\leq 1}(\mathbb{N})$ *is deterministic in the sense that for all* $t \geq 0$ *and bounded functions* $f : \mathbb{N} \to \mathbb{R}$

$$\mathbb{V}ar_x \left[\langle X(t),\, f \rangle \right] = 0\,. \tag{7.78}$$

Here, the quantity X *is the canonical coordinate process defined via*

$$X(t)(\omega) = \omega(t)\,. \tag{7.79}$$

In particular, for $A \subset [0,\, 1]^{\mathbb{N}}$*,*

$$\mathbb{P}_x(X(t) \in A) = \delta_{x(t)}(A)\,, \tag{7.80}$$

where $x(t)$ *is the solution to the coupled system of differential equations*

$$\begin{aligned} \frac{d}{dt} x_i(t) \quad = \quad & s(i-1)x_{i-1}(t)1_{\{i>1\}} - six_i(t) \\ & + d\binom{i+1}{2} x_{i+1}(t) - d\binom{i}{2} x_i(t) \\ & + (1-u(t))\alpha(t)x_1(t)1_{\{i=1\}} \\ & + c(1-u(t))((i+1)x_{i+1}(t) - ix_i(t)1_{\{i>1\}}) \\ & + cu(t)((i+1)x_{i+1}(t) - ix_i(t)) \\ & + (\alpha(t) + \gamma(t))(x_{i-1}(t)1_{\{i>1\}} - x_i(t))\,, \quad i \in \mathbb{N}\,, \end{aligned} \tag{7.81}$$

with the initial condition $x(0) = x$*. Here,*

$$u(t) = \sum_{i \geq 1} x_i(t)\,, \quad \alpha(t) = c\sum_{i \geq 2} ix_i(t)\,, \quad \gamma(t) = cx_1(t)\,, \tag{7.82}$$

and $u(t)$ *satisfies the differential equation*

$$\frac{d}{dt} u(t) = \alpha(t)u(t)(1-u(t)) - \gamma(t)u^2(t)\,. \tag{7.83}$$

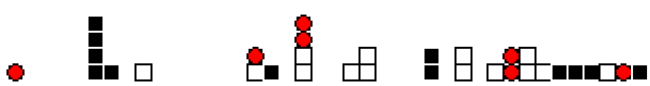
Proof. We can write $F_{f,g} = g \circ f$. Expression (7.78) is implied by the product rule (cf. Theorem II.2.14 of [TL1986] and its proof)

$$G(g_1 g_2 \circ f) = (g_1 \circ f)G(g_2 \circ f) + (g_2 \circ f)G(g_1 \circ f)\,. \tag{7.84}$$

The form of the equations (7.81) follows when inserting the function $f = e_i$ that takes the value 1 at i and is zero otherwise. □

Remark 7.19. *We claimed in the proof of Corollary 7.18 that the vanishing variance follows from the product rule* (7.84)*. At least in the case of bounded operators, this can be understood as follows: Let* $\mathcal{G}$ *be bounded and assume that its domain contains a separating algebra* $\mathcal{D}$*. Suppose that*

$$\mathcal{G}fg = f\mathcal{G}g + g\mathcal{G}f \tag{7.85}$$



holds for all $f,\, g \in \mathcal{D}$. *This implies for the associated Feller semigroup* T

$$
\begin{aligned}
T(t)fg &= \sum_{k\geq 0} \frac{\mathcal{G}^k}{k!} fg \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} (\mathcal{G}^m f)(\mathcal{G}^{k-m} g) \\
&= \left(\sum_{k\geq 0} \frac{\mathcal{G}^k}{k!} f \right) \left(\sum_{k\geq 0} \frac{\mathcal{G}^k}{k!} g \right) \\
&= (T(t)f)(T(t)g)\,.
\end{aligned}
\tag{7.86}
$$

The consequence is

$$
\mathbb{E}_x \left[f(X(t))^2 \right] = (T(t)f^2)(x) = [(T(t)f)(x)]^2 = \mathbb{E}_x \left[f(X(t)) \right]^2 \,. \tag{7.87}
$$

7.2.3 Step 3: The martingale problem and a related ODE

We have to show that the limiting martingale problem has at most one solution. We first recapitulate the notion of a martingale problem (cf. Chapter 4.3 of [EK1986]).

Definition 7.20 (The martingale problem)**.**
Let G *be as in Lemma 7.17 and let* $\mathcal{D}(G)$ *denote the domain of* G. *Let* $\mu \in \mathcal{M}_1(\mathcal{M}^2_{\leq 1}(\mathbb{N}))$. *Then, any* $D([0,\,\infty),\, \mathcal{M}^2_{\leq 1}(\mathbb{N}))$ *valued process* X solves the $(G,\, \mu)$ martingale problem *if*

$$
\mathcal{L}(X(0)) = \mu \tag{7.88}
$$

and for all $F \in \mathcal{D}(G)$ *the process*

$$
M^F(t) = F(X(t)) - F(X(0)) - \int_0^t (GF)(X(s))ds \tag{7.89}
$$

is a martingale with respect to the filtration $\{\mathcal{F}^X_t \;:\; t \geq 0\}$, *where*

$$
\mathcal{F}^X_t = \sigma(X(s) \;:\; s \leq t)\,. \tag{7.90}
$$

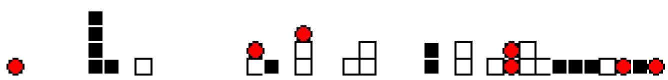
Since we already know that any solution is deterministic, it is equivalent to show, for any given initial value taken from the support of the initial condition μ, the coupled system of differential equation (7.18) is uniquely solvable. In order to apply standard results on differential equations, we introduce a Banach space for the $[0,\,1]^{\mathbb{N}}$ valued solution.

Definition 7.21 (state space of the differential equation)**.**
Define the measure $\nu \in \mathcal{M}(\mathbb{N}_0)$ *via its density*

$$
\nu(n) = 1 + n^2 \;,\; n \in \mathbb{N}_0\,. \tag{7.91}
$$

Let

$$
\begin{aligned}
\mathbb{B}_0 &= \left\{ a \in \mathbb{R}^{\mathbb{N}_0} \;:\; \sum_{j\geq 0} |a(j)|\nu(j) < \infty \right\}, \\
\mathbb{B}^2_0 &= \left\{ a \in \mathbb{R}^{\mathbb{N}_0} \;:\; \sum_{j\geq 0} |a(j)|(\nu(j))^2 < \infty \right\}
\end{aligned}
\tag{7.92}
$$



and

$$\begin{aligned}\mathbb{B} &= \left\{ a \in \mathbb{R}^{\mathbb{N}} \;:\; \sum_{j\geq 1} |a(j)|\nu(j) < \infty \right\} ,\\ \mathbb{B}^2 &= \left\{ a \in \mathbb{R}^{\mathbb{N}} \;:\; \sum_{j\geq 1} |a(j)|(\nu(j))^2 < \infty \right\} \end{aligned} \tag{7.93}$$

denote the sets of sequences indexed with $\mathbb{N}_0$ *and* $\mathbb{N}$ *respectively that have finite integral with respect to the measures* ν *and* ν^2 *respectively. Define on the respective sets the corresponding* L^1 *norms; e. g. on* $\mathbb{B}_0$ *the norm*

$$||\cdot|| = \sum_{j\geq 0} |(\cdot)(j)|\nu(j)\,. \tag{7.94}$$

We quickly quote the result that these sets are indeed Banach spaces.

Lemma 7.22. *The spaces* $\mathbb{B}_0$, $\mathbb{B}_0^2$, $\mathbb{B}$, $\mathbb{B}^2$ *are Banach spaces.*

Proof. This follows from the identification

$$\mathbb{B}_0 = L^1_\nu(\mathbb{N}_0) \tag{7.95}$$

which is known to be a Banach space, cf. e. g. Theorem III.6.6 of [DS1957]. It is not necessary to pass over to equivalence classes because the kernel of $||\cdot||$ is already trivial. □

Now, we can state the $\mathbb{B}_0$-valued differential equation that the Martingale problem will be related to. These equations are not identical to those seen in Corollary 7.18 because the total mass will be rescaled in order to obtain a probability vector.

Definition 7.23 (A related Banach valued differential equation)**.**
Consider for given $b_0 \in \mathbb{B}_0$ *and fixed* $T > 0$ *the Banach valued differential equation*

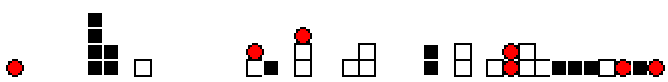
$$\begin{aligned} b \,:\, [0,T] &\to \mathbb{B}_0\,,\\ \tfrac{d}{dt}b(t) &= Q^*b(t) + H^*(b(t))\,,\\ b(0) &= b_0\,. \end{aligned} \tag{7.96}$$

Here, $Q^* \,:\, \mathbb{B}_0^2 \to \mathbb{B}_0$ *is the linear mapping given via*

$$\begin{aligned} (Q^*\beta)(0) &= 0\,,\\ (Q^*\beta)(j) &= s(j-1)\beta(j-1)1_{\{j>1\}} - sj\beta(j)\\ &\quad + \frac{d}{2}(j+1)j\beta(j+1) - \frac{d}{2}j(j-1)\beta(j)1_{\{j>1\}}\\ &\quad + c(j+1)\beta(j+1) - cj\beta(j)1_{\{j>1\}}\,, \end{aligned} \tag{7.97}$$

where $\beta \in \mathbb{B}_0^2$ *and* $j \in \mathbb{N}$*. The function* $H^* \,:\, \mathbb{B}_0 \to \mathbb{B}_0$ *is given via*

$$\begin{aligned} H^*(\beta)(0) &= \alpha(\beta)\beta(0)(1-\beta(0)) - \gamma(\beta)\beta(0)^2\,,\\ H^*(\beta)(j) &= \left[(1-\beta(0))\alpha(\beta) + c\beta(0)\right]\beta(1)1_{\{j=1\}}\\ &\quad + (\alpha(\beta)+\gamma(\beta))(\beta(j-1)1_{\{j>1\}} - \beta(j))\\ &\quad - \beta(j)(\alpha(\beta)(1-\beta(0)) - \gamma(\beta)\beta(0))\,, \end{aligned} \tag{7.98}$$



where $\beta \in \mathbb{B}_0$ and $j \in \mathbb{N}$. Here, the auxiliary quantities

$$\begin{aligned} \alpha(\beta) &= c \sum_{j \geq 2} j\beta(j) , \\ \gamma(\beta) &= c\beta(1) \end{aligned} \tag{7.99}$$

have been used.

The assertion is that it is sufficient to solve this Banach valued differential equation. Note that the arguments in Section 7.2.1 allow to conclude that any accumulation point μ of the sequence of initial conditions has also a finite fourth moment.

Lemma 7.24. *Let g be as in Definition 7.6 and let some $\mu \in \mathcal{M}_1(\mathcal{M}^4_{\leq 1}(\mathbb{N}))$ be given. Assume that there exists a unique solution to the equation (7.96) for all $T > 0$ and all*

$$b_0 \in \left\{ \tilde{b} \in \mathbb{B}_0^2 \, : \, \tilde{b}(i) \geq 0 \text{ for all } i \in \mathbb{N}_0, \, \tilde{b}(0) \in [0, 1], \, \sum_{i=1}^{\infty} \tilde{b}(i) = 1 \right\} . \tag{7.100}$$

Then, there exists at most one solution to the martingale problem (G, μ).

Proof. The set of initial conditions is rich enough because we can identify b_0 with the pair

$$(\psi(\mathbb{N}), \, \psi(\mathbb{N})^{-1}\psi) \in [0, 1] \times \mathcal{M}_1^4(\mathbb{N}) \tag{7.101}$$

for some $\psi \in \mathcal{M}^4_{\leq 1}(\mathbb{N})$ and vice versa. The assertion is that the system of differential equations in Definition 7.23 is the same as in Corollary 7.18 when the evolution is renormalized with the total mass. The normalized quantity evolves as follows:

$$\frac{d}{dt}\left(\frac{x_i(t)}{u(t)}\right) = \frac{1}{u(t)}\left(\frac{d}{dt}x_i(t)\right) - \left(\frac{x_i(t)}{u(t)}\right)\left(\frac{1}{u(t)}\frac{d}{dt}u(t)\right) . \tag{7.102}$$

Since the expression $\mathrm{d}/\mathrm{d}t \, x_i(t)$ is linear in the $\{x_j(t)\}$, we can for the first summand copy the evolution equations, replacing the $x_j(t)$ with their normalized pendants $u(t)^{-1}x_j(t)$. Corollary 7.18 finally states for the second summand that

$$\frac{1}{u(t)}\frac{d}{dt}u(t) = \alpha(t)(1 - u(t)) - \gamma(t)u(t) . \tag{7.103}$$

Renaming $u(t) = b(t)(0)$ and $u(t)^{-1}x_j(t) = b(t)(j)$ gives the system as stated in Definition 7.23. Uniqueness then carries over via the identification $b(t)(0) \cdot b(t)(j) = x_j(t)$ for $j \in \mathbb{N}$. $\square$

Remark 7.25. *The norm $|| \cdot ||$ on the Banach space $\mathbb{B}_0$ is chosen such that any vector of interest falls off quickly enough. The idea of this cut-off is to ensure that high indices do not play a decisive role and the Banach space is effectively some $\mathbb{R}^d$ with large d. We will struggle below with the fact that Q^* does not map $\mathbb{B}_0$ into itself (because it is expansive due to the j, j^2 terms). It remains an open question if there is a better choice for the underlying Banach space that for instance exploits the fact that Q^*b is built from difference expressions of neighboured components of b.*

The equations for b arise as the equations for Φ when this object is normalized to be a member of $\mathcal{M}_1(\mathbb{N})$. It will argued below in Lemma 7.32 that these are the equations for the limit of $\Psi^N(K^N)^{-1}$.

7.2.4 Step 4: Existence and uniqueness of the solution to the ODE

In order to apply Lemma 7.24, we have to show existence and uniqueness of the solution to the system of differential equations as introduced in Definition 7.23. Due to time constraints, we have to admit that the provided arguments are rather brief.

Proposition 7.26. *For arbitrary $b_0 \in \mathbb{B}_0^2$, there exists a unique classical solution to the system of differential equations as described in Definition 7.23.*

Proof. It is shown in the subsequent two Lemmas that the operator Q^* creates a strongly continuous semigroup on $\mathbb{B}_0$ when the initial values are taken from the smaller set $\mathbb{B}_0^2$ and that H^* is differentiable on the Banach space $\mathbb{B}_0$. The literal statement of Theorem 6.1.5 of [AP1983] (which is summarized in Appendix A.5) states existence and uniqueness of a classical solution if Q^* creates a strongly continuous semigroup on the whole space $\mathbb{B}_0$. We use the additional knowledge that even with the perturbation H^* the set $\mathbb{B}_0^2$ is never left by any possible solution, which is also shown in Lemma 7.27. We claim without further proof that the result of [AP1983] can be extended to this situation. □

In order to apply the mentioned results on Banach valued differential equations, we need to show that both the mappings Q^* and H^* in Definition 7.23 enjoy certain regularity properties.

Lemma 7.27. *Define the operator $Q^*_{\mathbb{N}}$ as the restriction of Q^* to the index set $\mathbb{N}$ instead of $\mathbb{N}_0$, i. e.*

$$Q^*_{\mathbb{N}} \pi^{\mathbb{N}_0}_{\mathbb{N}} b = Q^* b \qquad (b \in \mathbb{B}_0^2)\,, \tag{7.104}$$

where $\pi^{\mathbb{N}_0}_{\mathbb{N}}$ denotes the projection from $\mathbb{N}_0$ to $\mathbb{N}$. Then, the operator $Q^*_{\mathbb{N}}$ is the adjoint of the generator Q of a Feller Semigroup corresponding to a continuous time Markov Chain $X(t)$ with state space $\mathbb{N}$. Furthermore, for any $t \geq 0$,

$$\mathcal{L}\left[X(t)\right] \in \mathcal{M}_1^4(\mathbb{N})\,, \tag{7.105}$$

when the process starts in some arbitrary $\mathcal{L}\left[X(0)\right] \in \mathcal{M}_1^4(\mathbb{N})$. This remains true when the Markov process is modified by a nonlinear term that corresponds to the function H^*.

Proof. Identify the vector

$$b \in \mathbb{B} \cap \left\{ \tilde{b} \in \mathbb{R}_+^{\mathbb{N}} \,:\, \sum_{k \geq 1} \tilde{b}(k) \leq 1 \right\} \tag{7.106}$$

with the corresponding probability measure $\mu_b \in \mathcal{M}_{\leq 1}(\mathbb{N})$. Fix some $f \in C_b(\mathbb{N}, \mathbb{R})$ and calculate the action of the adjoint operator Q as follows:

$$\int_{\mathbb{N}} (Qf)(j)\, \mu_b(dj) = \int_{\mathbb{N}} f(j)\, (Q^*_{\mathbb{N}} \mu_b)(dj)\,. \tag{7.107}$$

In particular, choosing $\mu_b = \delta_{\{m\}}$, we obtain

$$\begin{aligned}
(Qf)(m) \quad &= \quad \sum_{j \geq 1} f(j) \Big[s(j-1)\delta_{\{m\}}(j-1)1_{\{j>1\}} - sj\delta_{\{m\}}(j) \\
&\qquad + \frac{d}{2}(j+1)j\delta_{\{m\}}(j+1) - \frac{d}{2}j(j-1)\delta_{\{m\}}(j)1_{\{j>1\}} \\
&\qquad + c(j+1)\delta_{\{m\}}(j+1) - cj\delta_{\{m\}}(j)1_{\{j>1\}} \Big] \\
&= \quad (f(m+1) - f(m))\, sm \\
&\qquad + (f(m-1) - f(m))\, d\binom{m}{2} 1_{\{m>1\}} \\
&\qquad + (f(m-1) - f(m))\, cm1_{\{m>1\}}\,.
\end{aligned} \tag{7.108}$$

This corresponds to the backward equations associated to a birth and death process X on $\mathbb{N}$; the operator $Q^*_{\mathbb{N}}$ thus generates the corresponding forward equations on $\mathcal{M}_1(\mathbb{N})$. Finiteness of the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} b(t) = Q^*_{\mathbb{N}} b(t) \tag{7.109}$$

in the norm of $\mathbb{B}_0^2$ follows thus when we can show that the fourth moment of the Markov process X remains finite.

The birth and death process describes a fixed colony in the Collision Free System where the emigration of the last particle is forbidden, just as in Proposition 2.6. The fourth moment is finite for all t; this can be seen like in Proposition 2.6 with the exception that the initial state does not necessarily lie stochastically below the equilibrium. Instead, we use that for any initial state that lies above the equilibrium the distribution is stochastically decreasing (this can for instance be compared with the Kingman coalescent that comes down from $+\infty$ at time $0+$ due to the quadratic death rate). The assertion then follows when assuming that the initial state already has finite fourth moment.

The modification due to the function H^* corresponds to additional time inhomogeneous birth and death rates that are proportional to the mean of the process. Still, the quadratic death rate ensures that the moments remain finite. □

We turn to the nonlinear term H^*.

Lemma 7.28. *The function H^* is Fréchet differentiable on $\mathbb{B}_0$.*

Proof. We show Gâteaux differentiability first. Define for $b,\, h \in \mathbb{B}_0$ the Gâteaux differential via the strong limit

$$\delta H^*(b,\, h) = \lim_{\eta \to 0} \frac{1}{\eta} \left(H^*(b + \eta h) - H^*(b) \right). \tag{7.110}$$

Since

$$\frac{1}{\eta} \left[\alpha(b + \eta h) - \alpha(b) \right] = \alpha(h)\,, \tag{7.111}$$

we define the limit candidate D via

$$\begin{aligned} D(0) \quad &= \quad \alpha(h) b(0)(1 - b(0)) + \alpha(b) h(0)(1 - b(0)) - \alpha(b) b(0) h(0) \\ &\quad\; -\gamma(h) b(0)^2 - 2\gamma(b) b(0) h(0)\,, \end{aligned} \tag{7.112}$$

and, for $j \geq 1$,

$$\begin{aligned} D(j) \quad = \quad & \left[-h(0)\alpha(b) + (1 - b(0))\alpha(h) + c h(0) \right] 1_{\{j=1\}} \\ & + (\alpha(h) + \gamma(h))(b(j-1) 1_{\{j>1\}} - b(j)) \\ & + (\alpha(b) + \gamma(b))(h(j-1) 1_{\{j>1\}} - h(j)) \\ & - h(j)(\alpha(b)(1 - b(0)) - \gamma(b) b(0)) \\ & - b(j)(\alpha(h)(1 - b(0)) - \alpha(b) h(0) - \gamma(h) b(0) - \gamma(b) h(0))\,. \end{aligned} \tag{7.113}$$

For given $\eta > 0$ we obtain thus

$$|| \frac{1}{\eta} (H^*(b + \eta h) - H^*(b)) - D || = \frac{1}{\eta} \sum_{j \geq 0} (j^2 + 1) | (H^*(b + \eta h) - H^*(b))(j) - \eta D(j) |\,. \tag{7.114}$$

This expression goes to zero for $\eta \to 0$ provided that the sum is finite. The latter is ensured by the fact that $||b||,\, ||h|| < \infty$: For instance, the j^{th} coordinate of $H^*(b + \eta h)$ ($j \geq 2$) is given via

$$\begin{aligned} H^*(b + \eta h)(j) \quad &= \quad (\alpha(b + \eta h) + \gamma(b + \eta h))((b + \eta h)(j-1) - (b + \eta h)(j)) \\ &\quad\;\; -(b + \eta h)(j)(\alpha(b + \eta h)(1 - (b + \eta h)(0)) \\ &\quad\;\; +(b + \eta h)(j)\gamma(b + \eta h)(b + \eta h)(0)) \\ &= \quad H_0^*(b,\, h)(j) + \eta H_1^*(b,\, h)(j) \\ &\quad\;\; +\eta^2 H_2^*(b,\, h)(j) + \eta^3 H_3^*(b,\, h)(j)\,. \end{aligned} \tag{7.115}$$

Here, we abbreviate

$$\begin{aligned}
H_0^*(b,\,h)(j) &= \pi(b)(b(j-1)-b(j)) \\
&\quad -b(j)\alpha(b)(1-b(0)) \\
&\quad +b(j)\gamma(b)b(0)\,, \\
H_1^*(b,\,h)(j) &= \pi(h)(b(j-1)-b(j))+\pi(b)(h(j-1)-h(j)) \\
&\quad -h(j)(\alpha(b)(1-b(0))-b(j)\alpha(h)(1-b(0))+b(j)\alpha(b)h(0) \\
&\quad +h(j)\gamma(b)b(0)+b(j)\gamma(h)b(0)+b(j)\gamma(b)h(0)\,, \\
H_2^*(b,\,h)(j) &= \pi(h)(h(j-1)-h(j)) \\
&\quad -h(j)\alpha(h)(1-b(0))+h(j)\alpha(b)h(0)+b(j)\alpha(h)h(0) \\
&\quad +b(j)\gamma(h)h(0)+h(j)\gamma(b)h(0)+h(j)\gamma(h)b(0)\,, \\
H_3^*(b,\,h)(j) &= h(j)\alpha(h)h(0) \\
&\quad +h(j)\gamma(h)h(0)\,;
\end{aligned} \tag{7.116}$$

and $\pi(\cdot)=\alpha(\cdot)+\gamma(\cdot)$. We can identify

$$H_0^*(b,\,h)(j)=H^*(b)(j) \qquad \text{and} \qquad H_1^*(b,\,h)(j)=D(j)\,, \tag{7.117}$$

and the claim follows from the bound

$$|H_2^*(b,\,h)(j)|,\ |H_3^*(b,\,h)(j)| \le \text{const}\cdot\max(|b(j-1)|,\,|h(j-1)|,\,|b(j)|,\,|h(j)|)\,, \tag{7.118}$$

where the constant depends on b and h, but no longer on j.

Moreover, the linear map $\delta H^*(b,\,\cdot)$ is continuous: Examining expressions (7.112) and (7.113), it follows that there exist constants $C(b)$, $\tilde{C}(b)$ such that

$$\begin{aligned}
\sup_{0\neq h\in\mathbb{B}_0}\frac{||\delta H^*(b,\,h)||}{||h||} &\le \sup_{0\neq h\in\mathbb{B}}\frac{C(b)}{||h||}\sum_{j\ge1}\nu(j)(|b|(j)+|b|(j-1))(|\pi(h)|+|h|(0)) \\
&\quad +\sup_{0\neq h\in\mathbb{B}}\frac{C(b)}{||h||}\sum_{j\ge1}\nu(j)(|h|(j)+|h|(j-1)) \\
&\le \sup_{0\neq h\in\mathbb{B}}\tilde{C}(b)\left(\sum_{j\ge0}\nu(j)(|b|(j)+|b|(j+1))\right) \\
&\qquad \cdot\frac{\left(\sum_{j\ge0}\nu(j)|h|(j)\right)}{||h||}\,.
\end{aligned} \tag{7.119}$$

The first sum is finite, since $||b||<\infty$; the second equals one by definition of the norm $||\cdot||$. Similarly, the map $b\mapsto\delta H^*(b,\,\cdot)$ as a map from $\mathbb{B}_0$ into the set of bounded linear operators on $\mathbb{B}_0$ is continuous with respect to the operator norm. This implies Fréchet differentiability, cf. Proposition 8 of Chapter IV of [GM1963]. □

7.2.5 Step 5: Uniqueness of the accumulation point of the initial conditions

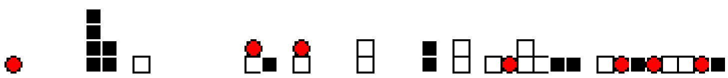
We consider the coupled systems $\{\zeta^{col,N}\ :\ N\in\mathbb{N}\}$ and omit the superscript *col* in the following. Assume that there are two accumulation points μ_1, μ_2 of the sequence (7.38) of initial conditions. Assume that the underlying probability space is modified in order to obtain almost sure convergence: There are two random variables

$$\Psi^{\infty,1},\,\Psi^{\infty,2} \quad \text{with} \quad \mathcal{L}\left[\Psi^{\infty,1}\right]=\mu_1 \quad \text{and} \quad \mathcal{L}\left[\Psi^{\infty,2}\right]=\mu_2 \tag{7.120}$$

such that

$$\begin{aligned}
\Psi^{\infty,1} &= \lim_{k\to\infty}\hat{\Psi}^{N_{1,k}}(t(N_{1,k}))\,, \\
\Psi^{\infty,2} &= \lim_{k\to\infty}\hat{\Psi}^{N_{2,k}}(t(N_{2,k}))
\end{aligned} \tag{7.121}$$

almost surely for different subsequences $\{N_{1,k}\}$, $\{N_{2,k}\}$.



Remark 7.29. *We observe that the modification of the probability space that ensures almost sure convergence along the subsequences goes beyond the usual Skorohod embedding. This is for the following reasons: The minor issue is to justify that both convergent subsequences can be defined on the same probability space. This is possible since the probability space is usually the unit interval. The major issue is that the processes ζ^N, $N \in \mathbb{N}$ are already coupled to a single realization of ζ; below, we will make use of the fact that both limit points converge towards the same realization of W for $t \to -\infty$. We thus have to make use of a conditional formulation of the Skorohod embedding which we do not specify here.*

The task is to prove the following Proposition.

Proposition 7.30. *Let the notation be as introduced above. Then, it follows necessarily that $\Psi^{\infty,1} = \Psi^{\infty,2}$.*

Before going into the proof, we introduce some further notation.

Definition 7.31 (Normalized hitting times and their limits)**.**
Define for given $\epsilon > 0$ the normalized hitting times

$$\begin{aligned} \hat{T}^N_{\epsilon N} &= T^N_{\epsilon N} - \frac{1}{\alpha}\log N\,, \\ \hat{T}_{\epsilon N} &= T_{\epsilon N} - \frac{1}{\alpha}\log N\,, \end{aligned} \tag{7.122}$$

the limiting hitting times

$$\begin{aligned} \tau^{\infty,1}(\epsilon) &= \lim_{k\to\infty} \hat{T}^{N_{1,k}}_{\epsilon N_{1,k}}\,, \\ \tau^{\infty,2}(\epsilon) &= \lim_{k\to\infty} \hat{T}^{N_{2,k}}_{\epsilon N_{2,k}} \end{aligned} \tag{7.123}$$

and the collision free limiting hitting time

$$\tau^{CF}(\epsilon) = \lim_{N\to\infty} \hat{T}_{\epsilon N} = \frac{1}{\alpha}\log\frac{\epsilon}{W}\,. \tag{7.124}$$

The superscript CF stands for the Collision Free System. Furthermore, let for $i \in \{1, 2\}$

$$\begin{aligned} u^i(t) &= b^i(t)(0)\,, \\ U^i(t) &= \pi^{\mathbb{N}_0}_{\mathbb{N}} b^i(t) \end{aligned} \tag{7.125}$$

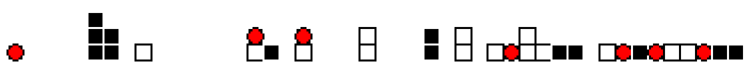
denote the proportion of occupied colonies *and the* proportional statistics *respectively. Here, b^i is the solution to the Banach valued differential equation as introduced in Definition 7.23 with the random initial condition given by $\Psi^{\infty,i}$, $i \in \{1, 2\}$.*

We quickly summarize that the introduced quantities are well-defined.

Lemma 7.32. *The limits in equations* (7.123) *and* (7.124) *exist and the pair $(u^i(t), U^i(t))$ satisfies*

$$\begin{aligned} u^i(t)U^i(t) &= \Phi(t)\,, \\ U^i(t) &= \lim_{k\to\infty} \frac{\Psi^{N_{i,k}}(t(N_{i,k})+t)}{K^{N_{i,k}}(t(N_{i,k})+t)}\,, \\ u^i(t) &= \lim_{k\to\infty} \frac{K^{N_{i,k}}(t(N_{i,k})+t)}{N_{i,k}}\,. \end{aligned} \tag{7.126}$$

Moreover, the quantities $\tau^{\infty,1}(\epsilon)$, $\tau^{\infty,2}(\epsilon)$, $\tau^{CF}(\epsilon)$ are continuous functions in ϵ and the quantities $u^i(t)$, $U^i(t)(j)$, $j \in \mathbb{N}$ are continuous in t.



Proof. The limit (7.124) follows from Corollary 2.11 and the existence of (7.123) because the limiting trajectory is continuous and monotone for small ϵ. For the same reason, the processes $\tau^{\infty,1}(\epsilon)$, $\tau^{\infty,2}(\epsilon)$ are continuous and monotonically increasing in ϵ.

The pair (u, U) is continuous since it satisfies the differential equation for b and the identities (7.126) follow from the definition of b. □

We turn to Proposition 7.30. The proof is split into two parts: Lemma 7.34 is the key result that identifies the limits of the evolutions (u^i, U^i) at $-\infty$ and shows that these do not depend on i. The final Lemma 7.35 then finishes the proof by stating that, given an initial value at $-\infty$, the possible trajectories are uniquely determined.

A preparatory result is that the normalized hitting times, considered as functions of ϵ, map to the whole negative time line.

Lemma 7.33. *For the normalized hitting times $\tau^{1,\infty}$, $\tau^{2,\infty}$, the following holds:*

$$\lim_{\epsilon\searrow 0} \tau^{i,\infty}(\epsilon) = -\infty\,, \quad i \in \{1, 2\}\,. \tag{7.127}$$

Proof. Recall that Corollary 4.14 allowed to bound

$$T^N_{\epsilon N} \le T_{\tilde{\epsilon} N} \tag{7.128}$$

for large N, where $\tilde{\epsilon}$ is a deterministic number that satisfies

$$\frac{\tilde{\epsilon}}{\epsilon} \to 1 \quad \text{for} \quad \epsilon \searrow 0. \tag{7.129}$$

Hence,

$$\begin{aligned}
\lim_{\epsilon\searrow 0} \tau^{i,\infty}(\epsilon) &= \lim_{\epsilon\searrow 0} \lim_{k\to\infty} \left(T^{N_{i,k}}_{\epsilon N_{i,k}} - \frac{1}{\alpha} \log N_{i,k} \right) \\
&\le \lim_{\epsilon\searrow 0} \lim_{k\to\infty} \hat{T}_{\tilde{\epsilon} N_{i,k}} \\
&= \lim_{\epsilon\searrow 0} \log \frac{\tilde{\epsilon}}{W}\,.
\end{aligned} \tag{7.130}$$

This together with the property $W > 0$ a. s. implies (7.127). □

The previous Lemma can be used to replace the limit $t \to -\infty$ of a continuous function $f(t)$ by the limit $\epsilon \to 0$ of $f(\tau^{i,\infty}(\epsilon))$. This is the idea of proof of the following Lemma that identifies the entrance laws of the triple (u, U, Φ) when the first component is normalized adequately.

Lemma 7.34. *For any $\epsilon > 0$ and fixed t_0, the limiting processes (u^i, U^i, Φ^i) can be extended up to time $-\infty$ and the obtained process satisfies*

$$\lim_{t\to-\infty} \left(\frac{u^i(t)}{e^{\alpha t}}, U^i(t), \Phi^i(t)\right) = (\exp(\alpha t_0) W, \Psi_\infty, 0)\,, \tag{7.131}$$

for $i \in \{1, 2\}$.

In order to tidy up the proof, it is split into three parts, each corresponding to one component of the triple in equation (7.131).

Proof of the convergence of the first component. We use

$$\begin{aligned}
\left| u^i(\tau^{i,\infty}(\epsilon)) - \epsilon \frac{\exp(\alpha \tau^{i,\infty}(\epsilon))}{\exp\left(\alpha (T^{N_{i,k}}_{\epsilon N_{i,k}} - t(N_{i,k}))\right)} \right| &\le \left| u^i(\tau^{i,\infty}(\epsilon)) - \frac{K^{N_{i,k}}(t(N_{i,k}) + \tau^{i,\infty}(\epsilon))}{N_{i,k}} \right| \\
&\quad + \left| \frac{K^{N_{i,k}}(T^{N_{i,k}}_{\epsilon N_{i,k}} + \eta(k))}{N_{i,k}} - \epsilon \exp(\alpha \eta(k)) \right|\,,
\end{aligned} \tag{7.132}$$

where we introduce the error term

$$\eta(k) = \tau^{i,\infty}(\epsilon) - (T^{N_{i,k}}_{\epsilon N_{i,k}} - t(N_{i,k}))\,. \tag{7.133}$$

In the limit $k \to \infty$, the first summand on the right hand side converges to zero because of the pointwise convergence towards u^i; the second converges to zero because of the path regularity properties of $K^N(\cdot)$ and the fact that

$$\eta(k) \to 0\,, \tag{7.134}$$

which follows from the convergence towards $\tau^{i,\infty}(\epsilon)$. Using as before (cf. Corollary 4.14) the bound

$$T_{\epsilon N_{i,k}} \leq T^{N_{i,k}}_{\epsilon N_{i,k}} \leq T_{\tilde{\epsilon} N_{i,k}}\,, \tag{7.135}$$

where $\tilde{\epsilon}\epsilon^{-1} \to 1$ in the limit $\epsilon \searrow 0$, we obtain

$$\begin{aligned}
\limsup_{\epsilon \searrow 0} \frac{u^i(\tau^{i,\infty}(\epsilon))}{\exp(\alpha(\tau^{i,\infty}(\epsilon))} &= \limsup_{\epsilon \searrow 0} \epsilon \lim_{k\to\infty} \exp\left(-\alpha(T^{N_{i,k}}_{\epsilon N_{i,k}} - t(N_{i,k}))\right) \\
&\leq \limsup_{\epsilon \searrow 0} \lim_{k\to\infty} \epsilon N_{i,k} \frac{(W + w(T^{N_{i,k}}_{\epsilon N_{i,k}}))}{\epsilon N_{i,k}} \exp(\alpha t_0) \\
&= W \exp(\alpha t_0)\,.
\end{aligned} \tag{7.136}$$

Using the second inequality in (7.135), we can also bound the lim inf from below with $W \exp(\alpha t_0)$. This together with Lemma 7.33 shows the claim. □

We turn to the second component.

Proof of the convergence of the second component. An estimate similar to (7.132) yields component-wise

$$U^i(\tau^{i,\infty}(\epsilon)) = \lim_{k\to\infty} \frac{\Psi^N(T^{N_{i,k}}_{\epsilon N_{i,k}})}{\epsilon N_{i,k}}\,. \tag{7.137}$$

By virtue of Theorem 4.1 and Corollary 4.14, we obtain for fixed $j \in \mathbb{N}$

$$\begin{aligned}
\limsup_{\epsilon \searrow 0} U^i(\tau^{i,\infty}(\epsilon))(j) &= \limsup_{\epsilon \searrow 0} \lim_{k\to\infty} \frac{\Psi^N(T^{N_{i,k}}_{\epsilon N_{i,k}})(j)}{\epsilon N_{i,k}} \\
&\leq \limsup_{\epsilon \searrow 0} \lim_{k\to\infty} \frac{(\Psi(T_{\tilde{\epsilon} N_{i,k}})(j) + C\tilde{\epsilon}^2 N_{i,k})}{\tilde{\epsilon} N_{i,k}} \frac{\tilde{\epsilon}}{\epsilon} \\
&= \limsup_{\epsilon \searrow 0} (\Psi_\infty + C\tilde{\epsilon}) \frac{\tilde{\epsilon}}{\epsilon}\,.
\end{aligned} \tag{7.138}$$

This implies the claim for the lim sup. The lim inf follows similarly, using the reversed bounds. □

Finally, we show the last assertion which is rather trivial.

Proof of the convergence of the third component. This claim follows from the exponential decay of u and the fact that

$$uU = \Phi\,. \tag{7.139}$$

□

Lemma 7.34 identified the limits at $-\infty$ of the evolutions and showed that these do not depend on the choice of accumulation point for the initial value. We are done when we can show that the evolutions are uniquely determined by these values at $-\infty$. For this, we copy the argument from part d) of Lemma 7.21 of [DG2010].

Lemma 7.35. *Fix some $A > 0$ and $B \in \mathcal{M}_1(\mathbb{N})$. Any two solutions (u_1, U_1), (u_2, U_2) of the system of differential equations satisfying*

$$\lim_{t\to-\infty} \exp(-\alpha t)u_i(t) = A\,, \quad \lim_{t\to-\infty} U_i(t) = B \qquad (i \in \{1, 2\}) \tag{7.140}$$

coincide.

Proof. Fix some A and assume that there are two nonnegative solutions u_1, u_2 to the system of differential equations both satisfying

$$\lim_{t\to-\infty} \frac{u_1(t)}{\exp(\alpha t)} = \lim_{t\to-\infty} \frac{u_2(t)}{\exp(\alpha t)} = A\,. \tag{7.141}$$

This implies

$$v(t) = |u_1(t) - u_2(t)| = o(\exp(\alpha t)\,, \quad (t \to \infty)\,. \tag{7.142}$$

In the limit $t \to -\infty$ we have for $i \in \{1, 2\}$

$$\frac{d}{dt}u_i(t) = \alpha_i(t)(1 - b_i(t)u_i(t))u_i(t) \tag{7.143}$$

where

$$b_i(t)u_i(t) = \left(1 + \frac{\gamma_i(t)}{\alpha_i(t)}\right)u_i(t) = O(\exp \alpha t)\,, \quad (t \to -\infty)\,. \tag{7.144}$$

Then, it follows that for a given time horizon $t \leq T_0$ we can find a constant $C(T_0)$ such that

$$|\alpha_i(t) - \alpha| \leq C(T_0)\exp(\alpha t) \tag{7.145}$$

and, for $t \leq T_0$,

$$\begin{aligned} \frac{d}{dt}v(t) &= |[\alpha_1(t)(1 - b_1(t)u(t))u_1(t)] \\ &\quad - [\alpha_2(t)(1 - b_2(t)u(t))u_2(t)]| \\ &\leq (\alpha + C(T_0)\exp(\alpha t))v(t)\,. \end{aligned} \tag{7.146}$$

Since the evolution v is zero at $-\infty$, it is greatest possible when the right hand side is attained. The differential equation for this upper bound is solved by

$$t \mapsto D\exp(\alpha t)\exp\left(\frac{C(T_0)}{\alpha}\exp \alpha t\right)\,, \tag{7.147}$$

where $D \in \mathbb{R}_+$ is a constant. Since by definition $v(t) = o(\exp(\alpha t)$ for $t \to -\infty$, only $D = 0$ is possible.

Uniqueness of U_i then follows since the functions α and γ are already determined given the evolution of u_i. □

This finishes the proof of Proposition 7.30 and thus the proof of Theorem 7.1.

Part IV

Appendix

We collect further material that is used in the main text.

A Results from the literature

A.1 A non-explosion criterion

In Section 2.1, the following non-explosion criterion is verified in order to show that both the N Colony System and the Collision Free System are well-defined. The criterion is taken from Theorem 4.3.6 of [DS2005].

Theorem A.1 (Non-explosion criterion)**.** *Let a Markov process with countable state space E and transition rates*

$$\{q(v,\ w)\ :\ v,w \in E\} \tag{A.1}$$

be given. Let R_v denote the rate at which state v is left, and $P_{v,w}$ the probability that state v is abandoned in favour of state w:

$$\begin{aligned} R_v &= \sum_{u\neq v} q(v,\ u)\,, \\ P_{v,w} &= \frac{q(v,\ w)}{\sum_{u\neq v} q(v,\ u)}\,. \end{aligned} \tag{A.2}$$

Suppose that there is some set family $\{F_M\}_{M\geq 1}$ satisfying

$$|F_M| < \infty\ ,\ F_M \subset F_{M+1}\ ,\ \bigcup_{M\geq 1} F_M = E\,. \tag{A.3}$$

Suppose further that there is a non-negative real function u on E satisfying

$$\lim_{M\to\infty} \inf_{v\notin F_M} u(v) = \infty\,, \tag{A.4}$$

and that there is some $\alpha \geq 0$ such that for any $v \in E$ with $R_v > 0$,

$$\sum_{w\in E} P_{v,w} u(w) \leq (1 + \frac{\alpha}{R_v}) u(v)\,. \tag{A.5}$$

Then, with probability one no explosion occurs.

A.2 Convergence of supercritical CMJ Processes

The following is a summary of the convergence results obtained by Nerman in his article [ON1981]. We use tildes on all random variables to distinguish the general results from our present context, but use the same symbols to indicate the connections. We restrict the description to the case where all particles live forever (in Nerman's notation, the life length λ is identically ∞).

Definition A.2 (The reproduction process)**.**
Let the reproduction process $\tilde{\xi}$ *be a point process on $\mathbb{R}_+$ and define the so called* $\tilde{\xi}$-measure

$$\tilde{\xi}\ :\ [0,\ \infty) \times \Omega \to \overline{\mathbb{N}} \tag{A.6}$$

and the so called reproduction function

$$\mu \ : \ [0,\ \infty) \to \overline{\mathbb{R}}_+ \tag{A.7}$$

via

$$\begin{aligned} \tilde{\xi}(t) &= \tilde{\xi}[0,\ t]\,, \\ \tilde{\mu}(t) &= \mathbb{E}\left[\tilde{\xi}(t)\right]\,. \end{aligned} \tag{A.8}$$

We state the CMJ dynamics verbally and refer to [ON1981] for the rigorous description.

Definition A.3 (The CMJ dynamics)**.**
Take a sequence $\{\tilde{\xi}^{(i)} \ : \ i \in \mathbb{N}_0\}$ *of i. i. d. copies of* $\tilde{\xi}$*. Assume that at time* $t = 0$ *there is one individual alive, and this individual creates new individuals according to the point process* $\tilde{\xi}^{(0)}$*: Assume that at any event of* $\tilde{\xi}^{(0)}$ *a birth takes place, i. e. a previously unused point process* $\tilde{\xi}^{(i)}$ *becomes active. From now on this point process also activates new processes, and so on. Let*

$$\tilde{K}(t) \tag{A.9}$$

denote the number of particles at time t, i. e. the number of active point processes $\tilde{\xi}^{(i)}$*.*

Since $\tilde{\mu}$ is monotonically increasing, one can define the Lebesgue-Stieltjes measure with respect to $\tilde{\mu}$ which is denoted with the same symbol. The following result is taken from Proposition 1.1, Corollary 3.3 and Theorem 5.4 of [ON1981].

Theorem A.4. *Suppose that* $\tilde{\xi}$ *and* $\tilde{\mu}$ *satisfy the following: the measure* $\tilde{\mu}$ *is not concentrated on a lattice, there exists some* $\tilde{\alpha} \in (0,\ \infty)$ *such that*

$$\int_0^\infty \exp(-\tilde{\alpha}t)\tilde{\mu}(dt) = 1\,, \tag{A.10}$$

and

$$\int_0^\infty t\exp(-\tilde{\alpha}t)\tilde{\mu}(dt) < \infty\,, \tag{A.11}$$

and finally

$$\mathbb{E}\left[\tilde{Z}\left[\log\tilde{Z}\right]^+\right] < \infty\,, \tag{A.12}$$

where

$$\tilde{Z} = \int_0^\infty \exp(-\tilde{\alpha}t)\tilde{\xi}(dt)\,. \tag{A.13}$$

Then,

$$\frac{\tilde{K}(t)}{\exp(\tilde{\alpha}t)} \to \tilde{W} \tag{A.14}$$

almost surely and in L^1*, where* $\tilde{W}$ *is a random variable that is almost surely positive.*

Bingham and Doney associate the moments of the normalized offspring $\tilde{Z}$ of one individual with the moments of $\tilde{W}$. The following is Theorem 1 of [BD1975].

Theorem A.5. *Let the notation be as above, and* $n \in \mathbb{N}$*,* $n \geq 2$*. Then,* $\tilde{W}$ *has finite* n^{th} *moment if and only if* $\tilde{Z}$ *has finite* n^{th} *moment.*

Ganuza and Durham finally show that existence of the second moment of $\tilde{W}$ is already sufficient for mean square convergence. The following is Theorem 1 of [GD1974].

Theorem A.6. *Let $\tilde{Z}$ be as in equation* (A.13)*. If* $\mathbb{E}\left[\tilde{Y}^2\right] < \infty$ *and additionally inequality* (A.11) *holds, then the convergence in* (A.14) *holds in mean square. Additionally,*

$$\mathbb{V}ar\left[\tilde{W}\right] = \frac{\mathbb{V}ar\left[\tilde{Y}\right]}{(1 - \int_0^\infty e^{-2\tilde{\alpha}t}\tilde{\mu}(dt))(\tilde{\alpha}\int_0^\infty te^{-\tilde{\alpha}t}\tilde{\mu}(dt))^2}\,. \tag{A.15}$$

Finally, Nerman shows almost sure convergence of the ratio of individuals of certain age amongst all particles alive. The following is Corollary 6.4 of [ON1981].

Theorem A.7. *Let $\tilde{\alpha}$ be as in Theorem A.4. Assume that there exists a $\tilde{\beta} \in (0,\, \tilde{\alpha})$ such that*

$$\int_0^\infty e^{-\tilde{\beta}t}\tilde{\mu}(dt) < \infty\,. \tag{A.16}$$

Let for $T > 0$

$$\tilde{K}^T(t) \tag{A.17}$$

denote the number of individuals having age less than or equal to T. Then,

$$\lim_{t\to\infty}\frac{\tilde{K}^T(t)}{\tilde{K}(t)} = 1 - e^{-\alpha T} \tag{A.18}$$

almost surely.

A.3 The topology of $\mathcal{M}_{\leq 1}(\mathbb{N})$

In this section, we collect facts about the set

$$\mathcal{M}_{\leq 1}(\mathbb{N}) = \{\psi \in [0,\, 1]^{\mathbb{N}} \;:\; \sum_{k\geq 1}\psi(k) \leq 1\} \tag{A.19}$$

that is endowed with the topology of weak convergence.

Proposition A.8. *The set $\mathcal{M}_{\leq 1}(\mathbb{N})$ with the Prohorov metric (taken with respect to the metric $\tilde{d}(m,\, n) = |m - n|$ on $\mathbb{N}$) is complete and separable.*

Proof. Cf. Theorem 3.1.7 of [EK1986]. □

In the following proposition, we propose a simpler metric on $\mathcal{M}_{\leq 1}(\mathbb{N})$.

Proposition A.9. *The metric d on $\mathcal{M}_{\leq 1}(\mathbb{N})$ that is defined via*

$$d(\mu,\, \nu) = \sum_{k\geq 1}|\mu(k) - \nu(k)| \tag{A.20}$$

makes the space complete. Convergence with respect to d is equivalent to weak convergence.

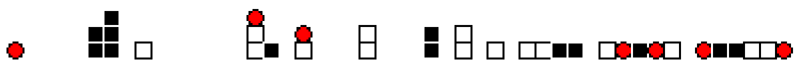
Remark A.10. *This allows to identify $\mathcal{M}_{\leq 1}(\mathbb{N})$ with a subset of the Banach space*

$$L^1_\lambda(\mathbb{N})\,, \tag{A.21}$$

where λ denotes the counting measure on $\mathbb{N}$.

Proof. Assume that $\mu_n \Rightarrow \mu$. Then, for fixed $\epsilon > 0$ and $M = M(\epsilon) \in \mathbb{N}$,

$$\sum_{k\geq 1}|\mu(k) - \mu_n(k)| \quad \leq \quad \sum_{k=1}^{M}|\mu(k) - \mu_n(k)| + \sum_{k>M}\mu(k) + \sum_{k>M}\mu_n(k)\,. \tag{A.22}$$



The first summand goes to zero because weak convergence implies point-wise convergence of the density function (take the bounded and continuous function $f = 1_{\{k\}}$) which in particular is uniform on the compact set $\{1, ..., M\}$. The second and the third summand are bounded by ϵ if M is large enough; the reason is that the sequences, taking values in some Polish space, are tight.

Assume now that $d(\mu_n, \mu) \to 0$. Then,

$$|\mu(\mathbb{N}) - \mu_n(\mathbb{N})| \leq \sum_{k\geq 1} |\mu(k) - \mu_n(k)| \,, \tag{A.23}$$

so that the total mass converges. Convergence of the tails follows similarly, so that this implies weak convergence. Completeness of the space follows via the embedding into the Banach space $L^1_\lambda(\mathbb{N})$ and the fact that the total mass is preserved. □

A.4 The convergence theorem from Ethier and Kurtz

The following is Corollary 8.16 of Chapter 4 of [EK1986] which itself relies on Corollary 8.12, Theorem 8.10 and Corollary 8.7 of the same source. The compact containment condition is stated in Remark 7.3 of Chapter 3.

Theorem A.11 (A criterion for weak convergence on path space)**.**
Let (E_n, d_n), (E, d) be complete and separable metric spaces ($n \in \mathbb{N}$). Assume the following:

1. *Let Y_n be a Markov process with right continuous sample paths in E_n corresponding to a measurable contraction semigroup $\{T_n(t) : t \geq 0\}$ with full generator $\hat{A}_n$. Let*

$$\eta_n : E_n \to E \tag{A.24}$$

be Borel measurable and define

$$X_n = \eta_n \circ Y_n \,. \tag{A.25}$$

2. *Let $A \subset C_b(E) \times C_b(E)$ and $\nu \in \mathcal{P}(E)$ and suppose that the $D([0, \infty), E)$ martingale problem for (A, ν) has at most one solution.*

3. *Suppose one of the following holds:*

   (a) *The linear span of $\mathcal{D}(A)$ contains an algebra that separates points. Furthermore, the processes $\{X_n : n \in \mathbb{N}\}$ satisfy a compact containment condition, i. e. for any $\epsilon > 0$ and $T > 0$ there exists a compact set $\Gamma_{\epsilon,T} \subset E$ such that*

$$\inf_{n\in\mathbb{N}} \mathbb{P}\left(X_n(t) \in \Gamma_{\epsilon,T} \text{ for all } t \in [0, T]\right) \geq 1 - \epsilon \,. \tag{A.26}$$

   (b) *The sequence $\{X_n\}$ is relatively compact.*

4. *The initial conditions converge, i. e.*

$$\mathcal{L}(X_n(0)) \Rightarrow \nu \,. \tag{A.27}$$

5. *The generators converge on a sufficiently large set: For each $(f, g) \in A$ and $T > 0$, there exists $(f_n, g_n) \in \hat{A}_n$ and sets $G_n \subset E_n$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(Y_n(t) \in G_n \text{ for all } t \in [0, T]\right) = 1 \tag{A.28}$$

and

$$\sup_{n\in\mathbb{N}} ||f_n|| < \infty \tag{A.29}$$

and moreover

$$\lim_{n\to\infty} \sup_{y\in G_n} |(f \circ \eta_n)(y) - f_n(y)| = \lim_{n\to\infty} \sup_{y\in G_n} |(g \circ \eta_n)(y) - g_n(y)| = 0 \,. \tag{A.30}$$

Then, there exists a solution X of the $D_E[0, \infty)$ martingale problem for (A, ν), and

$$X_n \Rightarrow X\,. \tag{A.31}$$

Remark A.12. *Conditions* (A.26) *and* (A.28) *are similar. In fact,* (A.26) *implies* (A.28) *(but not necessarily the additional assumption* (A.30)*). On the other hand,* (A.28) *implies* (A.26) *if the set G_n can be chosen compactly and if the probabilities in* (A.28) *can be made uniformly large by replacing the sets G_n with a fixed G_{n_0}.*

A.5 Lipschitz Perturbations of Linear Evolution Equations

The following results are taken from Chapter 6.1 of the book [AP1983].

Definition A.13 (Semilinear initial value problem)**.**
Let A be the infinitesimal generator of a C_0 semigroup $\{T(t) \,:\, t \geq 0\}$ on a Banach space $(X, \,||\cdot||)$ (that is, a semigroup of bounded linear operators satisfying

$$\lim_{t \searrow 0} T(t)x = x \tag{A.32}$$

in the norm topology for all $x \in X$). Let $\mathcal{D}(A)$ denote the domain of A. Let

$$f \;:\; X \to X \tag{A.33}$$

be a locally Lipschitz continuous function, i. e. we assume that for all $c > 0$ there is a constant $L(c)$ such that

$$||f(x) - f(y)|| \leq L(c)||x - y|| \tag{A.34}$$

for any $x, \; y \in X$ with $||x||, \; ||y|| < c$. Then, we consider for given $u_0 \in X$ the following initial value problem:

$$\begin{aligned} \frac{d}{dt}u(t) &= Au(t) + f(u(t))\;,\; t \geq 0\,; \\ u(0) &= u_0\,. \end{aligned} \tag{A.35}$$

There are different notions of solutions to (A.35). We collect the Definitions 4.2.2, 4.2.8 and 6.1.1 of [AP1983].

Definition A.14 (Notions of solution)**.**
For fixed $t_0 > 0$, there are the following notions of solutions on the time horizon $[0, \; t_0)$:

1. *A function $u \;:\; [0, \; t_0) \to X$ is a* classical solution *to* (A.35) *if $u(t) \in \mathcal{D}(A)$ for all $t \in (0, \; t_0)$, and if u is continuous and continuously differentiable on $[0, \; t_0)$ and the derivative satisfies* (A.35)*.*
2. *A function $u \;:\; [0, \; t_0) \to X$ is a* strong solution *to* (A.35) *if additionally the derivative u' is integrable with respect to the Lebesgue measure on the time interval $[0, \; t_0)$, i. e.*

$$\int_{[0, \; t_0)} ||u'(s)||\lambda(ds) < \infty\,. \tag{A.36}$$

3. *A continuous solution $(u(t) \;:\; t \in [0, t_0))$ to the integral equation*

$$u(t) = T(t)u_0 + \int_0^t T(t - s)f(u(s))ds \tag{A.37}$$

is a mild solution *to* (A.35)*.*

It is shown in the beginning of the chapters 4.2 and 6.1 of [AP1983] that any classical solution (and thus also any strong solution) is also a mild solution. The following statement is Theorem 6.1.2 and 6.1.5 of [AP1983].

Theorem A.15. *If f is continuously differentiable and $u_0 \in \mathcal{D}(A)$, then there exists a unique mild solution of the initial value problem* (A.35) *which is also a classical solution.*

B Proofs

In this section, proofs are collected that are excluded from the main text.

B.1 Continuation of the proof of Proposition 4.7

In Section 4.4, the asymptotic distribution of colouring events over the time line is considered. Proposition 4.7 stated that these events are clustered around the right end of the interval, namely around $A\lceil \epsilon N \rceil$. It remains to show the following:

Lemma B.1. *Let $C^N(a,\, b)$ denote the number of colourings between index a and b, $a,\ b \in \mathbb{N}$, where the i^{th} index is coloured with probability i/N.*

1. *There are no more than $(\log N)^2$ colourings between index $\left\lceil N^{\frac{1}{4}-\beta} \right\rceil$ and index $\left\lceil N^{\frac{1}{2}} \right\rceil$:*

$$\#C^N\left(\left\lceil N^{\frac{1}{4}-\beta}\right\rceil,\ \left\lceil N^{\frac{1}{2}}\right\rceil\right) \le (\log N)^2\,. \tag{B.1}$$

2. *There are no more than $2N^{1-2\delta}$ colourings between $\left\lceil N^{\frac{1}{2}} \right\rceil$ and $\lceil N^{1-\delta} \rceil$:*

$$\#C^N\left(\left\lceil N^{\frac{1}{2}}\right\rceil,\ \left\lceil N^{1-\delta}\right\rceil\right) \le 2N^{1-2\delta}\,. \tag{B.2}$$

   The last assertion remains true if $\delta(N)$ depends on N and is bounded away from $\frac{1}{2}$.

3. *There are no more than $2(A\epsilon)^2N$ colourings between $\lceil N^{1-\delta} \rceil$ and $A\lceil \epsilon N \rceil$:*

$$\#C^N\left(\left\lceil N^{1-\delta}\right\rceil,\ A\lceil \epsilon N\rceil\right) \le 2(A\epsilon)^2 N\,. \tag{B.3}$$

Proof of the first assertion. In order to obtain an upper bound, colour particles with probability

$$p = \frac{\left\lceil N^{\frac{1}{2}}\right\rceil}{N}\,, \tag{B.4}$$

and consider the colourings beginning with the first index 1 (instead of $N^{\frac{1}{4}-\beta}$).

Take a sequence of Bernoulli random variables $\{B_k(\cdot)\}$ with success probability $\cdot$, and apply Chernoff's Bound to an arbitrary function f to be specified below:

$$\begin{aligned}\mathbb{P}\left(\sum_{k=1}^{\left\lceil N^{\frac{1}{2}}\right\rceil} B_k(p) > f(N)\right) &\le \exp(-f(N)) \prod_{k=1}^{\left\lceil N^{\frac{1}{2}}\right\rceil} \mathbb{E}\left[\exp(-B_k(p))\right] \\ &= \exp(-f(N))\,(1+p(e-1))^{\left\lceil N^{\frac{1}{2}}\right\rceil}\,.\end{aligned} \tag{B.5}$$

Since the additional factor converges to $\exp(e-1)$, it remains to choose f such that $\exp(-f(N))$ is summable; and this is the case for $f(N) = (\log N)^2$, as can be seen using the following integral bound:

$$\int_e^\infty \exp(-[\log(x)]^2)dx \;=\; \int_1^\infty \exp(-y^2)\exp(y)dy < \infty\,. \tag{B.6}$$

□

Remark B.2. *a) One can also choose $f(N) = \log N \cdot \log\log N$, as can be seen via the estimate*

$$\begin{aligned}\int_e^\infty \exp(-\log(x)\log\log(x))dx &= \int_1^\infty \exp(-y\log(y))e^y dy \\ &= \int_1^\infty \exp(-y(\log(y)-1))dy \\ &\leq C_1 + \int_{C_2}^\infty \left(\frac{1}{y}\right)^2 dy\,, \end{aligned} \tag{B.7}$$

where we used some constants C_1 and C_2 and the bounds $\log(y) - 1 \geq \frac{1}{2}\log(y)$ and $y^{-\frac{1}{2}y} \leq y^{-2}$ for large enough y. On the other hand, choosing $f(N) = \log N$ would yield a harmonic integral which diverges.

b) Since

$$pN^{\frac{1}{2}} = 1 + o(1)\,, \tag{B.8}$$

we are in the regime of a Poisson approximation which leads to similar bounds, but we used Chernoff's Bound instead in order to avoid the question in how far the Poisson approximation gives uniform bounds.

Proof of the second assertion. The colouring probability is at most

$$p = \frac{\lceil N^{1-\delta}\rceil}{N} = \frac{1}{N^\delta} + o(1)\,. \tag{B.9}$$

Let the notation be as in the proof of the second assertion, and use again an exponential bound:

$$\begin{aligned}\mathbb{P}\left(\sum_{k=1}^{\lceil N^{1-\delta}\rceil} B_k(p) > f(N)\right) &\leq \exp(-f(N))\exp\left(\sum_{k=1}^{\lceil N^{1-\delta}\rceil}\log(1+p(e-1))\right) \\ &\leq \exp(-f(N))\exp\left((e-1)(N^{1-2\delta}+o(1))\right)\,. \end{aligned} \tag{B.10}$$

To make this summable, choose $f(N) = 2N^{1-2\delta}$. □

Remark B.3. *One can verify the Lyapunov Condition and thus replace the law of the random sum by a Normal distribution. This again leads to the same bounds; but the question remains open in how far this approximation works uniformly on $\mathbb{R}$ (although one can find Berry-Esseen type estimates for arrays in the literature, these are (just like in the standard formulation) absolute error bounds that are not useful in the tails far to the right).*

We consider the last time window.

Proof of the third assertion. Colour with probability $A\epsilon$. As before, we obtain

$$\mathbb{P}\left(C(1,\, A\lceil\epsilon N\rceil) \geq f(N)\right) \leq \exp(-f(N))\exp(A\lceil\epsilon N\rceil \cdot A\epsilon)\,,$$

and we may thus choose $f(N) = 2(A\epsilon)^2 N$. □

Remark B.4. *Finally, there is another property of the colouring evolution that has not been mentioned yet. If we do not make the replacement of the $s(i)$ by a deterministic approximation, we obtain for the number of colourings in the i^{th} index interval $\#C_i^{(N,\beta)}$ of length $N^{\frac{1}{2}-\beta}$ the following properties, where $\beta \in [0,\, \frac{1}{2})$ is fixed:*

$$\begin{aligned} 0 < \limsup_{N\to\infty} \sup_{1\leq i\leq i_{max}^{(N)}} \frac{\mathbb{E}\left[\#C_i^{(N,\beta)}\right]}{\mathbb{E}\left[s(1)\right] iN^{-2\beta}} &\leq 1\,, \\ 0 < \limsup_{N\to\infty} \sup_{1\leq i\leq i_{max}^{(N)}} \frac{\mathbb{V}ar\left[\#C_i^{(N,\beta)}\right]}{\mathbb{E}\left[s(1)\right] iN^{-2\beta} + \mathbb{V}ar\left[s(1)\right] i^2 N^{-4\beta}} &\leq 1\,. \end{aligned} \tag{B.11}$$

Here, $i_{max}^{(N)}$ denotes the number of index intervals, and is thus about $\epsilon N/N^{\frac{1}{2}-\beta}$. This last statement should be understood in the sense that one must divide the indices into blocks of length $\sqrt{N}$ in order to obtain nondegenerated numbers of colourings therein. We will not make use of this property, so we exclude the proof.

B.2 The inverse of the Q matrix of the harmonic random walk

In Section 5.4, the task arises to solve

$$\begin{pmatrix} 1 & -1 & & & & \\ -p & 1 & -q & & & \\ & -p & 1 & -q & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p & 1 & -q \\ & & & & -p & 1 \end{pmatrix} \begin{pmatrix} e(1) \\ e(2) \\ e(3) \\ \vdots \\ e(n-2) \\ e(n-1) \end{pmatrix} = \frac{1}{\rho c} \begin{pmatrix} 1 \\ 2^{-1} \\ 3^{-1} \\ \vdots \\ (n-2)^{-1} \\ (n-1)^{-1} \end{pmatrix}. \tag{B.12}$$

We invert the given matrix in the simple case $n = 4$ (which leads to a 3×3 matrix) and read off the general pattern below in Lemma B.5. Abbreviate $r = q^{-1}p$ and apply the Gauß algorithm:

$$\begin{array}{|ccc|ccc|}
\hline
1 & -1 & & 1 & & \\
-p & 1 & -q & & 1 & \\
 & -p & 1 & & & 1 \\
\hline
 & & & & & \\
\hline
1 & -1 & & 1 & & \\
 & q & -q & p & 1 & \\
 & -p & 1 & & & 1 \\
\hline
 & & & & & \\
\hline
1 & -1 & & 1 & & \\
 & 1 & -1 & r & \frac{1}{q} & \\
 & -p & 1 & & & 1 \\
\hline
 & & & & & \\
\hline
1 & -1 & & 1 & & \\
 & 1 & -1 & r & \frac{1}{q} & \\
 & & q & rp & r & 1 \\
\hline
 & & & & & \\
\hline
1 & -1 & & 1 & & \\
 & 1 & -1 & r & \frac{1}{q} & \\
 & & 1 & r^2 & \frac{r}{q} & \frac{1}{q} \\
\hline
 & & & & & \\
\hline
1 & -1 & & 1 & & \\
 & 1 & & r^2+r & \frac{r+1}{q} & \frac{1}{q} \\
 & & 1 & r^2 & \frac{r}{q} & \frac{1}{q} \\
\hline
 & & & & & \\
\hline
1 & & & r^2+r+1 & \frac{r+1}{q} & \frac{1}{q} \\
 & 1 & & r^2+r & \frac{r+1}{q} & \frac{1}{q} \\
 & & 1 & r^2 & \frac{r}{q} & \frac{1}{q} \\
\hline
\end{array} \tag{B.13}$$

Hence, the inverse is

$$\frac{1}{q}\begin{bmatrix} qr^2+qr+q & r+1 & 1 \\ qr^2+qr & r+1 & 1 \\ qr^2 & r & 1 \end{bmatrix} = \begin{bmatrix} r^2+r+1 & r+1 & 1 \\ r^2+r & r+1 & 1 \\ r^2 & r & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & \frac{1}{q} & \\ & & \frac{1}{q} \end{bmatrix}. \tag{B.14}$$

We can now read off the pattern of the inverse matrix when stopping at level $n+1$.

Lemma B.5. *Let $p \in [0,\ 1)$ and consider the $n \times n$ matrix $A^{(n)}$ with entries*

$$A^{(n)}_{11} = 1,\ A^{(n)}_{12} = -1,\ A^{(n)}_{1j} = 0,\ \text{if } j \geq 3\,, \tag{B.15}$$

and, for $i \geq 2$,

$$A^{(n)}_{ij} = \begin{cases} -p & ,\ \text{if } j = i-1 \\ 1 & ,\ \text{if } j = i \\ -(1-p) & ,\ \text{if } j = i+1 \\ 0 & ,\ \text{else}\,. \end{cases} \tag{B.16}$$

Define $q = 1 - p$ and

$$r = \frac{p}{q}\,. \tag{B.17}$$

Then, the inverse of $A^{(n)}$ is given by the product $R^{(n)}S^{(n)}$, where the factors have the following entries:

$$\begin{aligned} R^{(n)}_{ij} &= \sum_{k=(i-j)\vee 0}^{n-j} r^k\,, \\ S^{(n)}_{ij} &= \begin{cases} 1 & ,\ \text{if } i = j = 1 \\ \frac{1}{q} & ,\ \text{if } i = j > 1 \\ 0 & ,\ \text{else.} \end{cases} \end{aligned} \tag{B.18}$$

Proof. Let $i,\ j \in \{1,\ ...,\ n\}$ and assume $i \geq 2$. Then,

$$\begin{aligned} \left(A^{(n)}R^{(n)}\right)_{ij} &= \sum_k A^{(n)}_{ik} R^{(n)}_{kj} \\ &= \sum_k \left[\delta_{i,k} - \delta_{i,(k+1)}p - \delta_{i,(k-1)}q\right] \sum_{m=(k-j)\vee 0}^{n-j} r^m \\ &= \sum_{m=(i-j)\vee 0}^{n-j} r^m - p \sum_{m=(i-1-j)\vee 0}^{n-j} r^m - q \sum_{m=(i+1-j)\vee 0}^{n-j} r^m\,. \end{aligned} \tag{B.19}$$

If $i > j$,

$$\begin{aligned} \left(A^{(n)}R^{(n)}\right)_{ij} &= \sum_{m=i-j}^{n-j} r^m - p \sum_{m=i-1-j}^{n-j} r^m - q \sum_{m=i+1-j}^{n-j} r^m \\ &= -pr^{i-j-1} - pr^{i-j} + r^{i-j} \\ &= (1-q-p)r^{i-j} \\ &= 0\,, \end{aligned} \tag{B.20}$$

and if $i < j$,

$$\left(A^{(n)}R^{(n)}\right)_{ij} = \sum_{m=0}^{n-j} r^m - p\sum_{m=0}^{n-j} r^m - q\sum_{m=0}^{n-j} r^m = 0\,. \tag{B.21}$$

If $i = j$,

$$\left(A^{(n)}R^{(n)}\right)_{ii} = \sum_{m=0}^{n-i} r^m - p\sum_{m=0}^{n-i} r^m - q\sum_{m=1}^{n-i} r^m = q\,, \tag{B.22}$$

and this term cancels out when $S^{(n)}$ is multiplied to the right.

Similarly, when $i = 1$, we have

$$\left(A^{(n)}R^{(n)}\right)_{1j} = \sum_{m=(1-j)\wedge 0}^{n-j} r^m - \sum_{m=(2-j)\wedge 0}^{n-j} r^m\,, \tag{B.23}$$

and thus

$$\left(A^{(n)}R^{(n)}\right)_{11} = \sum_{m=0}^{n-1} r^m - \sum_{m=0}^{n-1} r^m + 1 = 1\,, \tag{B.24}$$

while

$$\left(A^{(n)}R^{(n)}\right)_{1j} = \sum_{m=0}^{n-j} r^m - \sum_{m=0}^{n-j} r^m = 0\,, \tag{B.25}$$

whenever $j > 1$. □

B.3 Continuation of the proof of Theorem 5.2

In order to emphasize the countability of the subset of $\mathcal{M}_{\text{fin}}(\mathbb{N})$ that supports the distributions of the statistics Ψ^N, Ψ, we introduce a new symbol.

Definition B.6 (The state space of Ψ and Ψ^N)**.**
Define the state space of the statistics

$$\mathbb{M} = \{\psi \in \mathcal{M}_{fin}(\mathbb{N}) \,:\, \psi(j) \in \mathbb{N}_0\}\ . \tag{B.26}$$

This countability is important because otherwise we had to work with an uncountable system of recurrence equations below which is not feasible. The task is to prove the following, which is the second assertion of Theorem 5.2:

Proposition B.7. *The variance of the hitting time of the harmonic random walk S is an upper bound for the conditional variance of the N Colony System:*

$$\begin{aligned}\sup_{\psi\in[\lceil\log N\rceil]} \mathbb{V}ar\left[T_{\epsilon N} - T_{\log N} \mid \Psi(T_{\log N}) = \psi\right] &\leq \mathbb{V}ar\left[T^S_{\epsilon N} - T^S_{\log N}\right]\,,\\ \sup_{\psi\in[\lceil\log N\rceil]} \mathbb{V}ar\left[T^N_{\epsilon N} - T^N_{\log N} \mid \Psi^N(T^N_{\log N}) = \psi\right] &\leq \mathbb{V}ar\left[T^S_{\epsilon N} - T^S_{\log N}\right]\,.\end{aligned} \tag{B.27}$$

Here, the sets $[i]$ are, for $i \in \mathbb{N}$, defined via

$$[i] = \{\psi \in \mathbb{M} \,:\, \sum_{k\geq 1} \psi(k) = i\}\,. \tag{B.28}$$

As already mentioned, the proof consists of four steps:

1. Rephrase the recurrence equations as given in Proposition 5.12 in terms of the N Colony System.
2. Generalize Proposition 5.12 in order to deal with more generally distributed holding times that may depend not only on the current but also on the next location.
3. Simplify the systems of equations by ignoring jumps that do not change the macrostate $[i]$ and show that the generalization of Proposition 5.12 is still applicable.
4. Simplify the system even further in order to reduce it to the harmonic random walk on $\mathbb{N}$. This is done by homogenisation of the waiting times and the step probabilities.

This is done in the following four subsections.

We need existence and uniqueness of solution to certain recurrence equations. Furthermore, we need monotonicity in the right hand side; that is if the waiting times increase, we want the solutions to increase. We thus cite the following Proposition from the book [JN1997] in advance.

Proposition B.8. *Let P be a transition matrix indexed with $\mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ is some countable set. Let $(Y_n)_{n \geq 0}$ be a discrete time Markov Chain evolving according to P. Assume that Y starts in point $i \in \mathbb{T}$ under the law $\mathbb{E}_i$. Assume that there are sets D, H such that*

$$\mathbb{T} = D \uplus H \,. \tag{B.29}$$

Let τ_H denote the hitting time of Y at H (this is 0 when starting in H). Let c be a non-negative vector indexed with $\mathbb{T}$ that is identically 0 on H. Consider the following system of linear equations, where x is the unknown:

$$\begin{aligned} [(id - P)x]_i &= c_i \;,\; i \in D \,, \\ x_i &= 0 \;,\; i \in H \,. \end{aligned} \tag{B.30}$$

Then, the following assertions hold:

1. *Existence: This system is solved by the so called potential ϕ, given by*

$$\phi_i = \mathbb{E}_i \left[\sum_{n=0}^{\tau_H - 1} c(Y_n) \right] \quad (i \in \mathbb{T}) \,, \tag{B.31}$$

   where the empty sum is defined to be zero.

2. *Uniqueness: If for all $i \in D$*

$$\mathbb{P}_i(\tau_H < \infty) = 1 \,, \tag{B.32}$$

   then there is at most one bounded solution.

3. *Monotonicity: If ψ is a non-negative vector that satisfies*

$$[(id - P)\psi]_i \geq c_i \;,\; i \in D \,, \tag{B.33}$$

   then $\psi_i \geq \phi_i$ for all $i \in \mathbb{T}$.

Proof. See [JN1997], Theorem 4.2.3, with the simplification that the "cost function" c is zero on the hitting boundary H. □

Remark B.9. *We took the formulation for a chain in discrete time; but the stated properties are properties of linear systems and carry thus over to the continuous time case when the right hand side c is modified correspondingly.*

B.3.1 Step 1: The recurrence equations associated to the N Colony System

In the present context, we ask for the variances of $T_{\epsilon N}$, $T_{\epsilon N}^N$ when starting in state $\psi \in [\lceil \log N \rceil]$. We introduce the following notation: For $\psi \in \mathbb{M}$, let

$$\Pi^{(1)}(\psi) = \sum_{k \geq 1} k \psi(k) \tag{B.34}$$

denote the number of particles associated to state ψ, let

$$\Pi^{(2)}(\psi) = \sum_{k \geq 2} \binom{k}{2} \psi(k) \tag{B.35}$$

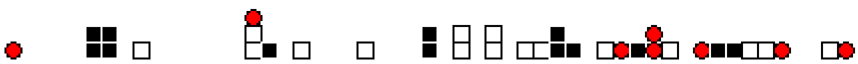
denote the number of interacting pairs, and let

$$K(\psi) = \sum_{k \geq 1} \psi(k) \tag{B.36}$$

denote the number of inhabited colonies. Moreover, let

$$T_k^{\pm} \psi \in \mathbb{M} \tag{B.37}$$

denote the state that arises from state ψ when one is added to, or subtracted from, the k^{th} component.



Proposition B.10. *Consider the N Colony System.*

1. *There exists some $\mathbb{M}\times\mathbb{M}$-indexed negated Q-matrix A and some function*

$$r \; : \; \mathbb{M} \to (0,\ \infty) \tag{B.38}$$

such that the vectors $(e(\psi))_\psi,\ (v(\psi))_\psi$

$$\begin{aligned} e(\psi) &= \mathbb{E}\left[T^N_{\epsilon N} \mid \Psi^N(0)=\psi\right] , \\ v(\psi) &= \mathbb{V}ar\left[T^N_{\epsilon N} \mid \Psi^N(0)=\psi\right] \end{aligned} \tag{B.39}$$

of conditional moments satisfy

$$\begin{aligned} Ae &= \left(\frac{1}{r(\psi)}\right)_{\psi\in\mathbb{M}} , \\ Av &= (f(\psi))_{\psi\in\mathbb{M}} , \end{aligned} \tag{B.40}$$

where

$$f(\psi) = \sum_{\phi\neq\psi} |A(\psi,\ \phi)| \mathbb{E}\left[\left(e(\phi)-e(\psi)+\frac{\gamma}{r(\psi)}\right)^2\right] . \tag{B.41}$$

Here, γ is an exponentially distributed random variable with mean 1.

2. *In particular, e and v are given by*

$$\begin{aligned} e &= A^{-1}\left(\frac{1}{r(\psi)}\right)_{\psi\in\mathbb{M}} , \\ v &= A^{-1}(f(\psi))_{\psi\in\mathbb{M}} , \end{aligned} \tag{B.42}$$

where A^{-1} is a positive linear operator that is single valued and monotone on the set

$$\mathcal{D} = \{c\in\mathbb{R}_+^{\mathbb{M}} \; : \; (Ac)_\psi < \infty \text{ for all } \psi\in\mathbb{M}\} . \tag{B.43}$$

In particular, $e,\ v\in\mathcal{D}$.

3. *Moreover, r can be specified as follows:*

$$\begin{aligned} r(\psi) &= c\frac{N-1}{N}\left[\Pi^{(1)}(\psi)-\psi(1)\right] + c\frac{K(\psi)-1}{N}\psi(1) \\ &\quad + s\Pi^{(1)}(\psi) + d\Pi^{(2)}(\psi) . \end{aligned} \tag{B.44}$$

Finally, A can be specified as follows: For $k\geq 1$,

$$\begin{aligned} A(\psi,\ \psi) &= 1 , \\ A(\psi,\ T^+_{k+1}T^-_k\psi) &= -\frac{1}{r(\psi)} s\psi(k)k , \\ A(\psi,\ T^-_{k+1}T^+_k\psi) &= -\frac{1}{r(\psi)} d\psi(k+1)\binom{k+1}{2} ; \end{aligned} \tag{B.45}$$

and $m\geq 1$

$$\begin{aligned} A(\psi,\ T^-_kT^+_1\psi) &= -\frac{1}{r(\psi)} c\frac{N-K(\psi)}{N}\psi(k)k , \\ A(\psi,\ T^-_kT^+_m\psi) &= -\frac{1}{r(\psi)} c\frac{\psi(m)-1_{\{k=m\}}}{N}\psi(k)k . \end{aligned} \tag{B.46}$$

Any other ϕ implies $A(\psi,\ \phi)=0$.

Proof. The function $r(\psi)$ is the rate at which state ψ is left. Similarly, A is the normalized Q-matrix of the process that is row-wise rescaled in order to set the diagonal elements to one and additionally multiplied with -1. But this matches exactly relations (5.44)-(5.46) of Proposition 5.12.

That the operator A^{-1} is well-behaved can be seen as follows: The hitting time of the embedded discrete time jump chain τ_H can be bounded above by the hitting time of a random walk; this was shown in the proof of the first assertion, cf. Section 5.2. This also implies that the vector $e(\psi)$ is uniformly bounded. The same holds for the second moment; and thus also for the variance. The assertion then follows from Proposition B.8. □

Apparently, there is no hope to solve these linear systems explicitly. Below, we will simplify the system in order to obtain approximate solutions.

B.3.2 Step 2: General Variance equations

In Step 3 below, the equations for the N Colony System as obtained in Proposition B.10 will be simplified; the idea will be to ignore transitions that leave K^N unchanged. In order to justify this, we will need the following technical extension of Proposition 5.12. The setting is now more general; holding times are not anymore necessarily exponentially distributed and may depend not only on the current state but also on the subsequent one.

Proposition B.11. *Let $(Y_n)_{n\geq 0}$ be a discrete time Markov chain with state space $\mathbb{T}$ and transition matrix π (where $\pi(s,\ s) \neq 0$ is allowed). Assume that, under $\mathbb{P}_i$, Y_n starts in state i. Let $(\gamma_{s,t}^{(n)})_{n\geq 0, s,t\in\mathbb{T}}$ be independent random variables that are independent of $(Y_n)_{n\geq 0}$, and interpret $\gamma_{s,t}^{(n)}$ as the waiting time between the n^{th} and $(n+1)^{th}$ jump when $Y_n = s$ and when $Y_{n+1} = t$. Assume that*

$$\gamma_{s,t}^{(n)} \overset{d}{=} \gamma_{s,t}^{(m)} \tag{B.47}$$

for all $n,\ m \in \mathbb{N}$ and $s,\ t \in \mathbb{T}$.

Define the process $(X_t)_{t\geq 0}$ via

$$X_t = Y_{J_t}\,, \tag{B.48}$$

where

$$J_t = \sup\{n \geq 0 \ :\ \sum_{k=0}^{n-1} \gamma_{Y_k, Y_{k+1}}^{(k)} \leq t\}\,. \tag{B.49}$$

For $H \subset \mathbb{T}$ consider the hitting time

$$\tau_H = \inf\{t \geq 0 \ :\ X_t \in H\}\,. \tag{B.50}$$

Abbreviate expectation and variance of τ_H with

$$\begin{aligned} e(i) &= \mathbb{E}_i\left[\tau_H\right]\,, \\ v(i) &= \mathbb{V}ar_i\left[\tau_H\right]\,. \end{aligned} \tag{B.51}$$

Then, the vector $(e_i)_{i\in\mathbb{S}}$ is a solution to the following system of equations:

$$\begin{aligned} e(i) - \sum_j \pi(i,\ j) e(j) &= \mathbb{E}_i\left[\gamma_{i,Y_1}^{(0)}\right]\ ,\ \text{if } i \notin H\,; \\ e(i) &= 0\ ,\ \text{if } i \in H\,. \end{aligned} \tag{B.52}$$

Similarly, the vector $(v_i)_{i\in\mathbb{S}}$ satisfies

$$\begin{aligned} v(i) &= \sum_{j\neq i} \pi(i,\ j) v(j) + f(i)\,,\ \text{if } i \notin H\,, \\ v(i) &= 0\ ,\ \text{if } i \in H\,; \end{aligned} \tag{B.53}$$

where

$$f(i) = \sum_j \pi(i,\ j)\mathbb{E}\left[\left(e(j) - e(i) + \gamma_{i,j}^{(0)}\right)^2\right] . \tag{B.54}$$

Proof. Since still

$$\tau_H = \sum_{k=0}^{\tilde{\tau}_H - 1} \gamma^{(k)}_{Y_k, Y_{k+1}} , \tag{B.55}$$

where $\tilde{\tau}_H$ is the hitting time of $(Y_n)_{n\geq 0}$ at H, stationarity yields

$$\mathbb{E}_i\left[\tau_H - \gamma^{(0)}_{i,Y_1} \mid Y_1 = j\right] = \mathbb{E}_j\left[\tau_H\right] . \tag{B.56}$$

From here onwards the proof is identical to the proof of Proposition 5.12. □

B.3.3 Step 3: Approximation using a reduced system

We introduce for the N Colony System a reduced system

$$(R^N(t))_{t\geq 0} \tag{B.57}$$

that ignores changes in the microstates but that still fits into the framework of Proposition B.11. In particular, the embedded discrete time jump process will be Markovian and the moments of the waiting times will be similar to those of the exponential distribution. This system is a harmonic random walk (for which we calculated the moments in Section 5.4) except for inhomogeneities in the step probabilities and the holding times. In the final step below these inhomogeneities are removed. This will finish the proof.

Assume that the process is in state $\psi \in [i]$. For each $\phi \in [i-1] \cup [i+1]$ there is a certain probability

$$P^{(i)}(\psi,\ \phi) \tag{B.58}$$

to be in microstate ϕ when the corresponding macrostate is reached. For fixed ψ, the vector $(P^{(i)}(\psi,\ \phi))_\phi$ sums to one. The reduced system R^N now is obtained by merging all paths

$$\psi \in [i] \to \psi_1 \in [i] \to \psi_2 \in [i] \to \ldots \to \psi_k \in [i] \to \phi \notin [i] \tag{B.59}$$

into

$$\psi \in [i] \to \phi \notin [i] . \tag{B.60}$$

In other words, if $R^N(t) = \psi \in [i]$, the next jump goes to $\phi \notin [i]$ according to the transition kernel $P^{(i)}$. The embedded jump chain is therefore Markovian while the waiting times are mixtures of sums of exponentials. It will turn out that - due to the embedded process ζ^S - the concrete shape of the kernel $P^{(i)}$ does not matter.

We rephrase Proposition B.10.

Lemma B.12. *Consider the reduced system with the transition kernels $P^{(i)}$ as introduced above. Let as before U_i^N denote the time when K^N leaves state i for the first time. Then, mean and variance of the hitting time of the N Colony System satisfy*

$$\begin{aligned} Be &= b\,, \\ Bv &= g\,, \end{aligned} \tag{B.61}$$

where

$$\begin{aligned} b(\psi) &= \mathbb{E}\left[U_i^N - T_i^N \mid \Psi^N(T_i^N) = \psi\right] , \\ g(\psi) &= \sum_{\phi \neq \psi} |B(\psi,\ \phi)| \cdot \\ &\quad \mathbb{E}\left[\left(e(\phi) - e(\psi) + U_i^N - T_i^N\right)^2 \mid \Psi^N(T_i^N) = \psi,\ \Psi^N(U_i^N) = \phi\right] , \end{aligned} \tag{B.62}$$

if $\psi \in [i]$. Here, B is the normalized Q matrix that has the following entries:

$$\begin{aligned} B_{\psi,\psi} &= 1\,, \\ B_{\psi,\phi} &= -P^{(i)}(\psi,\ \phi) \qquad \textit{for all } \phi \in [i-1] \cup [i+1]\,, \\ B_{\psi,\phi} &= 0 \qquad \textit{for all } \phi \notin [i-1] \cup \{\psi\} \cup [i+1]\,. \end{aligned} \tag{B.63}$$

Proof. The hitting times of the original and the reduced systems have the same distributions (this can be seen by coupling). Using Proposition B.11, the moments of the hitting times of the reduced system satisfy the given equations; this is implied by the following identity:

$$\mathbb{E}\left[\mathbb{E}\left[U_i^N - T_i^N \mid \Psi^N(T_i^N) = \psi,\ \Psi^N(U_i^N) = \phi\right] \mid \Psi^N(T_i^N) = \psi\right] = \\ \mathbb{E}\left[U_i^N - T_i^N \mid \Psi^N(T_i^N) = \psi\right]\,. \tag{B.64}$$

□

We wish to simplify the right hand sides of the given linear systems. Propositions B.8 and B.10 state that the solutions are monotone in the right hand side; so we may insert the upper bounds that were obtained in Lemma 5.7 by comparison with system ζ^S. This leads to the following corollary.

Corollary B.13. *Any solution (e, v) to the system that is obtained when replacing the conditional moments in Lemma B.12 by*

$$\frac{1}{\rho c i},\ \frac{2}{(\rho c i)^2} \tag{B.65}$$

are upper bounds for the exact solutions.

This corollary removes the inhomogeneities in the waiting times. It remains to relate the embedded jump chain to the random walk. This is done in the following last step.

B.3.4 Step 4: Reduction to a random walk

Recall that, before introducing the reduced system $(R^N(t))_{t\geq 0}$, the i^{th} macrostate row of the normalized Q matrix of the N Colony System looked schematically as follows (we assume for simplicity that there are only $\#[i] = 3$ microstates):

$$\begin{array}{cc|ccc|ccc|ccc|cc} \cdots & 0 & * & * & * & 1 & * & * & * & * & * & 0 & \cdots \\ \cdots & 0 & * & * & * & * & 1 & * & * & * & * & 0 & \cdots \\ \cdots & 0 & * & * & * & * & * & 1 & * & * & * & 0 & \cdots \end{array} \tag{B.66}$$

The first block corresponds to state $[i-1]$, the second to state $[i]$ and the third to state $[i+1]$. Each $*$ is non-positive, and each row sums to 0. In the Collision Free System, the block $[i-1]$ would be identically zero, because this is the probability to jump from state $[i]$ to state $[i-1]$.

After the collapse, the i^{th} row of the obtained matrix B as introduced in Lemma B.12 looks as follows:

$$\begin{array}{cc|ccc|ccc|ccc|cc} \cdots & 0 & * & * & * & 1 & 0 & 0 & * & * & * & 0 & \cdots \\ \cdots & 0 & * & * & * & 0 & 1 & 0 & * & * & * & 0 & \cdots \\ \cdots & 0 & * & * & * & 0 & 0 & 1 & * & * & * & 0 & \cdots \end{array} \tag{B.67}$$

Again, the left and the right block sum row-wise to -1. In the Collision Free System, the left block is still identically zero, and due to the transformation, the right block is a (negative) stochastic matrix; in the N Colony System however, in general neither of the two starred blocks will be stochastic on their own.

We aim to make these blocks stochastic in order to make use of the following simplifications: We can take advantage of the fact that the right hand sides of the linear systems (B.61) do not depend on the microstate $\psi \in [i]$ (this was stated in Corollary B.13). We can thus make the a priori assumption that the (approximated) solutions e, v also do not depend on the microstate; this implies that the vectors e, v projected on the coordinate range $[i]$ are multiples of the vector $(\ldots, 1, 1, 1, \ldots)$. These vectors in turn are right-eigenvectors to stochastic matrices. In effect, this procedure will allow to break down the infinite system of equations to a finite system that is indexed only with the macrostates. This is done and justified in Corollary B.16 below.

First, we must perform the transformation of the blocks in (B.67) to stochastic matrices. The idea is to shift row-wise mass from the right to the left; we need to argue algebraically that such a shift does not cause a strong perturbation in the solution.

Lemma B.14. *Let $\mathcal{D} \subset \mathbb{R}^{\mathbb{M}}$ be as in Corollary B.10. Let R be an $\mathbb{M} \times \mathbb{M}$ indexed matrix such that $-R$ is a Q matrix. Assume that $r_{\psi,\psi} = 1$ for all ψ, and that the operator R^{-1} is positive, single valued and monotone on the domain $\mathcal{D}$. Construct a matrix $\tilde{R}$ as follows: For each $\psi \in \mathbb{M}$, multiply the half row right to the diagonal element $r_{\psi,\psi}$ with some $m_\psi^- \le 1$ and the half row left to the diagonal element with some $m_\psi^+ \ge 1$ such that the row still sums to 0.*

Fix some non-negative $\mathbb{M}$-indexed vector c that lies in the domain of R^{-1}. Assume that there is a finite solution $\tilde{e}$ to the linear system

$$\tilde{R}\tilde{e} = c \tag{B.68}$$

that does not depend on the microstates and is decreasing in the macrostate. Then, the (unique) solution to

$$Re = c \tag{B.69}$$

is component-wise dominated by $\tilde{e}$.

Proof. Fix $\psi \in [i]$. Assume that only the ψ^{th} row is modified, i. e.

$$\tilde{R} = R + M \tag{B.70}$$

where M is zero except for the ψ^{th} row. This row sums to 0, and the entries $(m_{\psi,\phi})_{\phi\in[i-1]}$ to the left of the diagonal have negative signs while the entries $(m_{\psi,\phi})_{\phi\in[i+1]}$ to the right are positive (the modulus increases to the left, but due to the sign convention the change is negative). Then,

$$R\tilde{e} = (\tilde{R} - M)\tilde{e} = c - M\tilde{e}\,, \tag{B.71}$$

and consequently

$$\tilde{e} = R^{-1}\left(c - M\tilde{e}\right) = e + R^{-1}\left(-M\tilde{e}\right)\,. \tag{B.72}$$

Finally, $-M\tilde{e}$ is non-negative and lies in the domain of R^{-1}, when $\tilde{e}$ is as described above:

$$\begin{aligned}
\left(M\tilde{e}\right)_\psi &= \left[\tilde{e}_{[i-1]} \sum_{\phi\in[i-1]} m_{\psi,\phi} + \tilde{e}_{[i+1]} \sum_{\phi\in[i+1]} m_{\psi,\phi}\right] \\
&= -\left[\tilde{e}_{[i-1]} - \tilde{e}_{[i+1]}\right] \sum_{\phi\in[i-1]} |m_{\psi,\phi}| \;\in (-\infty,\; 0]\,.
\end{aligned} \tag{B.73}$$

□

We now have to make sure that not too much mass is shifted to the left (if one creates thereby a drift that points downwards it is not reasonable for the original claim to hold).

Corollary B.15. *Fix for $\psi \in [i]$ some constants m_ψ^-, m_ψ^+ and $p_i \in (0,\ 1)$, $i \in \mathbb{N}$. Assume that the i^{th} (macrostate) row of the matrix B in Lemma B.12 is, via the procedure described in Lemma B.14, transformed into the matrix $\tilde{B}_i$ that is given by*

$$\begin{array}{cc|ccc|ccc|ccc|cc} \cdots & 0 & & \vdots & & 1 & 0 & 0 & & \vdots & & 0 & \cdots \\ \cdots & 0 & \cdots & -p_i \cdot P^{(i,L)} & \cdots & 0 & 1 & 0 & \cdots & -(1-p_i)P^{(i,R)} & \cdots & 0 & \cdots \\ \cdots & 0 & & \vdots & & 0 & 0 & 1 & & \vdots & & 0 & \cdots \end{array} \tag{B.74}$$

where both $P^{(i,L)}$ and $P^{(2,R)}$ are stochastic matrices. Then, it is possible to do this such that

$$p \equiv p_i < \frac{1}{2}\,. \tag{B.75}$$

Proof. In the N Colony System, the greatest mass to the left side of the diagonal is obtained via

$$\sup_{\psi \in [i]} \sum_{\phi \in [i-1]} |B(\psi,\ \phi)| = \sup_{\psi \in [i]} \ \mathbb{P}\left(\Psi^N(U_i^N) \in [i-1] \mid \Psi^N(T_i^N) = \psi\right)\,. \tag{B.76}$$

This is smaller than $\frac{1}{2}$ by virtue of Lemma 5.9 and the embedding of ζ^S into ζ^N. □

We finally show that the collected approximations lead to a system of equations that corresponds to an asymmetric random walk that is reflected at 1.

Corollary B.16. *Let the matrix $\tilde{B}$ be as in Corollary B.15. Consider the set of macrostate indices*

$$M = \{1,\ 2,\ ...,\ \lceil \epsilon N \rceil - 1\} \tag{B.77}$$

and an $\mathbb{M}$ indexed nonnegative vector b. Assume that b is constant amongst all microstates associated to a given macrostate, i. e. for all $i \in M$ there exists a number b_i such that for all $\psi \in [i]$

$$b_\psi = b_i\,. \tag{B.78}$$

Then, a solution to the infinite system

$$\tilde{B}e = b\,, \tag{B.79}$$

where the i^{th} macrostate row of $\tilde{B}$ consists of the matrix $\tilde{B}_i$ as defined in Lemma B.15, is given by the vector e defined by

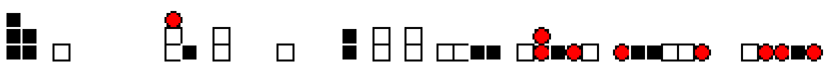
$$e_\psi := e_i^{fin}\,, \tag{B.80}$$

when $\psi \in [i]$. Here, e^{fin} is the solution to the finite system

$$\begin{pmatrix} 1 & 1 & & & & \\ p & 1 & -q & & & \\ & \ddots & \ddots & \ddots & & \\ & & p & 1 & -q & \\ & & & p & 1 & -q \\ & & & & p & 1 \end{pmatrix} \begin{pmatrix} e_1^{fin} \\ e_2^{fin} \\ \vdots \\ e_{|M|-2}^{fin} \\ e_{|M|-1}^{fin} \\ e_{|M|}^{fin} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{|M|-2} \\ b_{|M|-1} \\ b_{|M|} \end{pmatrix}\,, \tag{B.81}$$

where p is as in Corollary B.15 and $q = 1 - p$.

Proof. Let $n = |M|$. Due to the reduction discussed above, the system of equations is transformed into the following one:



$$\begin{pmatrix} 1 & 1 & & & & & \\ -pP^{(2,L)} & 1 & -qP^{(2,R)} & & & & \\ & & \ddots & \ddots & & & \\ & & -pP^{(n-2,L)} & 1 & -qP^{(n-2,R)} & & \\ & & & -pP^{(n-1,L)} & 1 & -qP^{(n-1,R)} \\ & & & & -pP^{(n,L)} & 1 \end{pmatrix} \begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \\ \underline{e}_{n-2} \\ \underline{e}_{n-1} \\ \underline{e}_n \end{pmatrix} = \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_{n-2} \\ \underline{b}_{n-1} \\ \underline{b}_n \end{pmatrix}. \tag{B.82}$$

Here, 1 is the identity matrix indexed with the microstates, $P^{(i,L)}$, $P^{(i,R)}$ denote the stochastic matrices obtained in Corollary B.15 and $\underline{e}_i$, $\underline{b}_i$ are vectors, again indexed with the microstates; and, by assumption, $(\underline{b}_i)_\psi = b_i$ for all $\psi \in [i]$.

Recall that for any stochastic matrix the vector $(..., 1, 1, 1, ...)^T$ is (right-)eigenvector with eigenvalue 1. The i^{th} (macrostate) row of the system reads thus as follows:

$$-p\underline{e}_{i-1} + \underline{e}_i - q\underline{e}_{i+1} = \underline{b}_i\,. \tag{B.83}$$

Since by assumption $(\underline{b}_i)_\psi \equiv b_i$ for all i and all $\psi \in [i]$, any solution e that is constant amongst the microstates solves (B.81) and vice versa. □

This finishes the reduction of the infinite dimensional system of recurrence equations related to the N Colony System to that of the harmonic random walk. Since the moments of the hitting times of the latter have already been calculated in Section 5.4, the proof of Theorem 5.2 is complete.

B.4 The proof of Corollary 7.13

The task is to prove the following statement from Section 7.2.1.

Corollary B.17. *The sequence*

$$\mathcal{L}\left[\left(\hat{\Psi}^N(t(N)+t)\right)_{t\geq 0}\right] \tag{B.84}$$

of measures on $D([0,\infty), \mathcal{M}_{\leq 1}(\mathbb{N}))$ is relatively compact.

We use the following tightness criterion for not necessarily $\mathcal{M}_{\leq 1}(\mathbb{N})$-valued càdlàg processes. This is Corollary 7.4 of Chapter 3 of [EK1986].

Theorem B.18. *Let (E, r) be complete and separable, and let $\{X_n\}$ be a sequence of processes with sample paths in $D([0,\infty), E)$. Then $\{X_n\}$ is relatively compact if and only if the following two conditions hold:*

a) For every $\eta > 0$ and rational $t \geq 0$, there exists a compact set $\Gamma_{\eta,t} \subset E$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left(X_n(t) \in \Gamma_{\eta,t}\right) \geq 1 - \eta\,. \tag{B.85}$$

b) For every $\eta > 0$ and $T > 0$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left(w'(X_n, \delta, T) \geq \eta\right) \leq \eta\,. \tag{B.86}$$

Here, the modulus of continuity w' is used, which is for some path $x \in D([0,\infty), E)$ defined as follows:

$$w'(x, \delta, T) = \inf_{\{t_i\}} \max_i \sup_{s,\,t\in[t_{i-1},\,t_i)} d(x(s), x(t))\,. \tag{B.87}$$

The infimum is taken over all partitions of the form

$$0 = t_0 < t_1 < \dots < t_{n-1} < T \leq t_n \tag{B.88}$$

such that $n \in \mathbb{N}$ *and*

$$\min_{i \leq n}(t_i - t_{i-1}) > \delta\,. \tag{B.89}$$

We are now ready to prove the corollary.

Proof. Relative compactness follows from tightness since $D([0,\,\infty),\,\mathcal{M}_{\leq 1}(\mathbb{N}))$ is Polish. For tightness, we check the conditions of Theorem B.18. Condition a) which states that the time marginals are tight follows immediately from Corollary 7.12. We turn to condition b) which is a path regularity property. Namely, we have to find for given $\eta > 0$ and $T > 0$ some $\delta > 0$ such that

$$\limsup_{N\to\infty} \mathbb{P}\left(w'(\hat{\Psi}^N,\,\delta,\,T) \geq \eta\right) \leq \eta\,, \tag{B.90}$$

where w' is the modulus of continuity, cf. expression (B.87). We take the following metric on $\mathcal{M}_{\leq 1}(\mathbb{N})$:

$$d(\mu,\,\nu) = \sum_{k\geq 1} |\mu(k) - \nu(k)|\,. \tag{B.91}$$

This is justified in Appendix A.3. We now use Lemma 7.11 to bound the transition rates; this allows to dominate w' with the modulus of continuity of a Poisson process with rate $O(N)$ and jumps of size $O(N^{-1})$.

The details are as follows: Choose some B such that inequality (7.41) holds for $\eta/2$. On the complement of this set, we can use the fact that the transition rates of $\hat{\Psi}^N$ are on $[0,\,T]$ bounded by

$$NB(\frac{d}{2} + s + c) = N\tilde{B}\,, \tag{B.92}$$

where $\tilde{B}$ is some constant. Using the fact that, in the metric $d(\cdot,\,\cdot)$, the jump heights of $\hat{\Psi}^N$ are bounded by $4N^{-1}$, this leads to

$$\mathbb{P}\left(w'(\hat{\Psi}^N,\,\delta,\,T) \geq \eta\right) \leq \mathbb{P}\left(w'(\frac{4}{N}P_{N\tilde{B}},\,\delta,\,T) \geq \eta\right) + \frac{\eta}{2}\,, \tag{B.93}$$

where $P_{N\tilde{B}}$ is a Poisson process of rate $N\tilde{B}$. This object can now be bounded by taking the partition into intervals of length 2δ:

$$\begin{aligned}
\mathbb{P}\left(w'(\frac{1}{N}P_{N\tilde{B}},\,\delta,\,T) \geq \frac{\eta}{4}\right) &\leq \mathbb{P}\left(\max_{1\leq i\leq \lceil \frac{T}{2\delta}\rceil} P_{N\tilde{B}}([t_{i-1},\,t_i)) \geq \frac{N\eta}{4}\right) \\
&= 1 - \left[\mathbb{P}\left(P_{N\tilde{B}}([0,\,2\delta)) < \frac{N\eta}{4}\right)\right]^{\lceil \frac{T}{2\delta}\rceil} \\
&= 1 - \left[\mathbb{P}\left(\sum_{k=1}^{\lceil 4^{-1}N\eta\rceil} \gamma_k > N2\delta\tilde{B}\right)\right]^{\lceil \frac{T}{2\delta}\rceil}\,.
\end{aligned} \tag{B.94}$$

Here, $\{\gamma_k\}$ is a sequence of i. i. d. exponentially distributed unit mean variables. Referring to the strong law of large numbers, δ can be made small enough such that the term in brackets converges towards 1 for $N \to \infty$. □

C Index of notation and references

Throughout the text, the following notation is used.

$\mathbb{Z}$	The set of integers, $\{..., -2, -1, 0, 1, 2, ...\}$
$\mathbb{N}$	The set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{N}_0$	The set of nonnegative integers, $\{0, 1, 2, 3, ...\}$
$\mathbb{R}$	The set of real numbers, $(-\infty, \infty)$
$\mathbb{R}_+$	The set of nonnegative real numbers, $[0, \infty)$
$\mathcal{M}(E)$	The set of Borel measures on the Polish space E
$\mathcal{M}_{\text{fin}}(E)$	The set of finite Borel measures on E, $\mathcal{M}_{\text{fin}}(E) \subset \mathcal{M}(E)$
$\mathcal{M}_{\leq 1}(E)$	The set of finite Borel measures with total mass less or equal to one
$\mathcal{M}_1(E)$	The set of Probability measures on E
$D(I, E)$	The set of E valued càdlàg functions on the time interval I
$\mathcal{B}(E)$	The Borel sigma field on E
$\uplus$	Disjoint union
$\lfloor\cdot\rfloor$	The floor function, $\sup\{n \in \mathbb{Z} : n \leq \cdot\}$
$\lceil\cdot\rceil$	The ceiling function, $\inf\{n \in \mathbb{Z} : n \geq \cdot\}$
$\wedge$	The minimum of two quantities
$\vee$	The maximum of two quantities
s, c, d	The constants of birth, migration, and death respectively

The following symbols refer to processes and functionals.

ζ, ζ^{col}	The Collision Free System, cf. Definitions 1.3 and 3.1
$\zeta^N, \zeta^{col,N}$	The N Colony System, cf. Definitions 1.1 and 3.1
$\mathbb{S}, \mathbb{S}^N$	The state spaces of ζ and ζ^N, cf. Definitions 1.3 and 1.1
Ψ, Ψ^N	The statistics of the respective systems, cf. Definition 1.5
K, K^N	The number of inhabited colonies in the respective systems, cf. Definition 1.7
Π, Π^N	The number of particles in the respective systems, cf. Definition 1.7
$\mathbb{M}$	The state space of Ψ and Ψ^N, cf. Definition B.6
$T_{\epsilon N}, T^N_{\epsilon N}$	The hitting times at level $\lceil \epsilon N \rceil$, cf. Definition 1.8
α	The growth constant of K: $K(t) = O(\exp(\alpha t))$ for $t \to \infty$, cf. Theorem 2.1
W	The random growth limit of K: $K(t)\exp(-\alpha t) \to W$ for $t \to \infty$, cf. Theorem 2.1
w	The random fluctuation of W: $K(t)\exp(-\alpha t) = W + w(t)$, cf. Remark 2.2
Ψ_∞	The limit of $\Psi(t)/K(t)$ for $t \to \infty$, cf. Theorem 2.1

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Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Hilfsmittel angefertigt habe.

Frank Schirmeier,

Erlangen, den 4. 3. 2010.